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Avoiding triples in arithmetic progression*

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Some patterns cannot be avoided ad infinitum. A well-known example of such a pattern is an arithmetic progression in partitions of natural numbers. We observed that in order to avoid arithmetic progressions, other patterns emerge. A visualization is presented that reveals these patterns. We capitalize on the observed patterns by constructing techniques to avoid arithmetic progressions.

More formally, van der Waerden numbers $W(k, l)$ express the smallest n such that partitioning $\{1, \ldots, n\}$ into k sets yields at least one set containing an arithmetic progression of length at least l. Computing these numbers for $l > 2$ is a very hard combinatorial problem. We focus on avoiding triples ($l = 3$) in arithmetic progressions. We guide the search procedure by transforming observed symmetries in best known lower bounds into additional constraints. Using our method, several lower bounds for van der Waerden numbers have been improved. As a consequence, a new pattern also emerges between the best known lower bounds. We conclude with open problems regarding van der Waerden numbers as well as some bold conjectures that challenges existing work on the subject.

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1. Introduction

Ramsey Theory offers several patterns that cannot be avoided ad infinitum. One such pattern is arithmetic progression while partitioning the natural numbers. More formally, for every k, l there exists a smallest n such that any partition of $\{1, \ldots, n\}$ into k color classes, at-least-one color class contains an arithmetic progression of length l. In 1927, Bartel Leendert van der Waerden proved this theorem [23]. For a given k and l the smallest n is known as the van der Waerden number $W(k, l)$.

After almost a century, only seven van der Waerden numbers are known with $k \geq 2, l \geq 3$. All of them, as well as the largest known lower bounds, have been determined by computation. The smallest ones have been es-

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tablished in the 1970's: $W(2,3) = 9$, $W(2,4) = 35$, and $W(3,3) = 27$ by Chvátal [6]; $W(2,5) = 178$ by Stevens and Shantaram [21]; and $W(4,3) = 76$ by Beeler and O'Neil [4]. Around the same time, a lower bound generator was developed by Rabung [18]. More recent progress regarding van der Waerden numbers was achieved by encoding the problem as a Boolean satisfiability problem (SAT) and using the power of SAT solvers to obtain solutions of the resulting propositional formulas [7]. Using years of computational effort, Kouril was able to establish $W(2,6) = 1132$ [16] and $W(3,4) = 293$ [14]. We generalized the method of Rabung to improve several lower bounds [10]. Rabung and Lotts, in turn, generalized our method [19].

Solving math problems using SAT technology has been successful in Ramsey Theory and beyond. Apart from the results mentioned above, SAT has been used for off-diagonal van der Waerden numbers (i.e., a variant such that the arithmetic progression restriction differs per color class) [1, 2] and Green-Tao numbers [17]. Successes outside Ramsey theory include solving the Erdős discrepancy problem [13] and Boolean Pythagorean triples problem [11].

Considerable effort has also been invested into establishing upper bounds of the van der Waerden numbers. The original proof by van der Waerden bounded the numbers above by an Ackermann function in l. Such a function grows faster than any primitive recursive function. Since the proof of Shelah [20] in 1986, the van der Waerden numbers are known to be bound above by a primitive recursive function. Gowers [9] has tightened these upper bounds even more by providing an alternative proof of the Szemerédi theorem [22] on arithmetic progressions. A stronger upper bound for $l = 3$ was found by Bourgain [5].

Currently there exists a great gap between methods that determine when arithmetic progression is inevitable (upper bounds) and those that try to avoid it (lower bounds). Only when a lower bound meets the corresponding upper bound, a new van der Waerden number is established. This paper contributes to existing work by introducing novel techniques to improve lower bounds.

On the bright side, it appears that while avoiding one pattern —in this case arithmetic progression— other patterns emerge. A van der Waerden *certificate* $W(k,l,n)$ is a partition of $\{1,\ldots,n\}$ into k color classes such that no color class contains an arithmetic progression of length l. We refer to a largest possible certificate $W(k,l,W(k,l)-1)$ as an extreme certificate. For all known $W(k,l)$, there exists an extreme certificate showing clear patterns. In this paper, we capitalized on observed patterns to improve lower bounds for several unknown $W(k,l)$ – in particular those with $l = 3$.

We focus on exploiting patterns called *internal symmetries* [12] and the novel pattern *pre-partitioning*. An internal symmetry maps a set of assignments onto itself. In the context of van der Waerden numbers, the set of assignments represents a certificate. Many patterns observed in extreme certificates can be expressed as internal symmetries. In earlier work [12], we constructed two improved lower bounds by forcing internal symmetries. Here, we show that this concept can be used to further push forward the current state of the art. Pre-partitioning, which groups elements based on the primitive root of the length of patterns, is also effective in improving lower bounds. As most recent works, we translate the problem into SAT together with constraints enforcing internal symmetries. This translation is small, quite natural and very effective. After adding these constraints, the computational costs to find certificates is reduced by several orders of magnitude.

The remainder of this paper is structured as follows: Section 2 describes the known van der Waerden numbers, the largest known lower bounds and a technique to visualize van der Waerden certificates. Section 3 presents the current methods to construct lower bounds based on certificates. The concept of pre-partitioning and internal symmetries and how to apply these patterns in the context of van der Waerden numbers is explained in Section 4. The experimental results are presented in Section 5. Finally, we post several open problems in Section 6 and draw some conclusions in Section 7.

2. Preliminaries

2.1. Lower bounds of van der Waerden numbers

Van der Waerden numbers, first introduced by van der Waerden [23], arise from the following theorem:

Theorem 2.1 (van der Waerden)**.** *Given two positive integers k and l, there exists a smallest number $W(k,l)$ with the following property:*

For each partition $\{1,2,\ldots,n_0\} = C_1 \cup C_2 \cup \ldots \cup C_k$ (with $n_0 \geq W(k,l)$) there is at-least-one color class C_s with $s \in \{1,\ldots,k\}$ which contains an arithmetic progression of length at least l.

An arithmetic progression of length l is a progression of numbers $a, a+d, a+2d, \ldots, a+(l-1)d$ for some $d > 0$.

Definition 2.1. *A van der Waerden certificate $W(k,l,n)$ is a partition of the elements $\{1,2,\ldots,n\}$ into k color classes C_s with $s \in \{1,\ldots,k\}$, such that no color class contains an arithmetic progression of length $\geq l$.*

The latter is equivalent to stating that $W(k,l) > n$. A certificate $W(k,l,n)$ therefore provides a lower bound n for the van der Waerden number $W(k,l)$. We refer to an *extreme certificate* $W(k,l,n)$ if $n = W(k,l) - 1$.

Only seven small van der Waerden numbers have been established. These van der Waerden numbers, as well as the best known lower bounds and their sources, are summarized in Table 1.

Table 1: Known van der Waerden numbers and the best known lower bounds. A * after some citations indicate that the lower bound can be computed using the method described in the referenced article, but that the article does not mention the bound. These omissions are likely the result of the computational costs to determine these bounds

$l \backslash k$	*2*	*3*	*4*	*5*	*6*
3	9 [6]	27 [6]	76 [4]	$>$ 125 [7]	$>$ 207 [18]
4	35 [6]	293 [14]	$>$ 1048 [18]	$>$ 2254 [18]	$>$ 9778 [18]
5	178 [21]	$>$ 965 [18]	$>$ 17705 [10]	$>$ 24045 [18]	$>$ 63473 [10]
6	1132 [16]	$>$ 8886 [18]	$>$ 91331 [10]	$>$ 246956 [18]*	$>$ 816981 [10]*
7	$>$ 3703 [18]	$>$ 43855 [18]*	$>$ 420216 [10]*		
8	$>$ 11495 [10]	$>$ 238400 [18]*			
9	$>$ 41265 [10]				

2.2. Visualization of certificates

Most of the ideas presented in this paper are motivated by patterns that were observed in the extreme and largest known certificates of van der Waerden numbers. As these patterns are hard to discover in the plain certificates, we developed a visualization technique [10] to make the patterns easier to spot. A similar visualization technique was used by John Venn to show that the decimals of the number π are random [24]. In contrast, we will show that the extreme and largest known certificates of van der Waerden numbers are *not* random.

The visualization can be used for certificates $W(k,l,n)$ for $k \geq 3$. Each color class C_s with $s \in \{1,\ldots,k\}$ is assigned an oriented edge. Color class C_1 is assigned the horizontal edge $\longleftarrow$. The other color classes C_s are assigned an oriented edge $\longleftarrow$ rotated by $360(s-1)/k$ degrees clockwise. We start drawing from an arbitrary starting point and loop through the elements in the certificate in increasing order. For each element we draw the oriented edge of the color class it occurs in. Each next edge is drawn starting from the point where its predecessor ended. An example certificate is visualized in Figure 1.

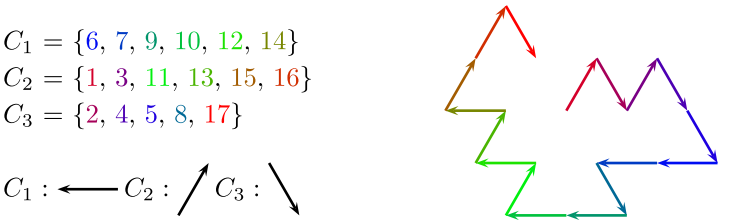

Figure 1: Example of the visualization of a certificate. For each element an oriented edge is drawn depending on the color class it occurs in. The colors represent the value of each element.

Additionally, the edges are colored to provide some intuition about the corresponding element. The edge representing the first element is colored red. The color of each next edge is gradually changed compared to the color of its predecessor. The colors change from red to blue to green and back to red. In Figure 1, each element has the same color as the corresponding edge. Because the arrowheads are useful for explanation, but confusing in practice, they are omitted in the actual visualizations.

3. Generators and satisfiability

The most common pattern that can be observed in extreme and largest known van der Waerden certificates is *repetition*. For a certificate $W(k, l, n)$ this pattern is as follows. Let $m = \lfloor \frac{n}{l-1} \rfloor$. For each element $1 \leq i \leq m(l-1)$, it holds that the elements i and $i+m$ occur in the same color class. So given a certificate $W(k, l, m)$, we can construct a certificate $W^*(k, l, m(l-1))$ by applying (with $C_s \in W(k, l, m)$ and $C_s^* \in W^*(k, l, m(l-1))$:

$$i \in C_s \Rightarrow i + jm \in C_s^* \qquad i \in C, j \in \{0, \ldots, l-2\}. \tag{1}$$

Throughout the paper, certificates $W(k, l, n)$ will be constructed and computed that contain this repetition. Also, m will denote the size of the base certificate and equals $\lfloor \frac{n}{l-1} \rfloor$. For all $k \geq 2$, $l \geq 3$ —except $k \in \{2, 3, 5\}$ and $l = 3$[1]— there exists an extreme or largest known certificate of size $m(l-1)+1$. We will use 0 for the additional element which usually can be placed in color class C_1. By using 0 instead of $n+1$, the techniques below are more natural to explain. Notice that an arithmetic-free partition

[1]In Section 4.2, we present an improved lower bound for $k = 5$, $l = 3$ and also the new bound has the exception that element 0 cannot be added to C_1.

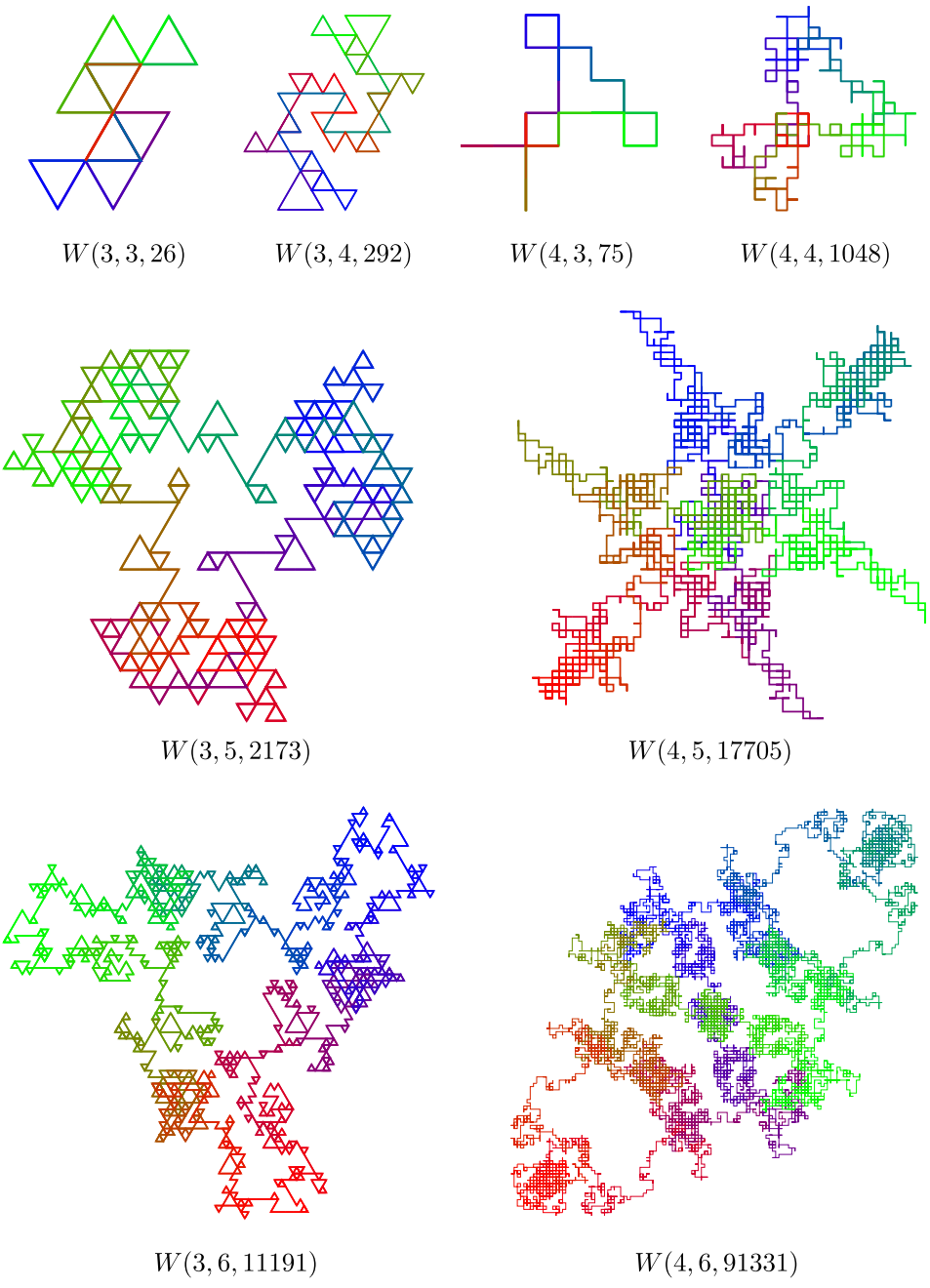

Figure 2: Visualizations of some extreme or largest known certificates.

of $\{0,\ldots,n\}$ can be easily transformed to an arithmetic-free partition of $\{1,\ldots,n+1\}$ by replacing each element i by $i+1$.

Below we present three techniques to search for certificates. First, the method by Rabung [18], presented in Section 3.1, constructs certificates $W(k,l,p(l-1)+1)$ with p prime. Second, the Zipping method [10], presented in Section 3.2, generalizes the first method to use it for non-prime numbers. Third, we show in Section 3.3 how certificates can be computed by translating the problem into SAT [7].

3.1. Power residue coloring

Rabung [18] proposed a method called *power residue coloring* to construct certificates of the form $W(k,l,p(l-1)+1)$ with p prime. In the first step of this method the primitive root of unity r_n of n is computed. Throughout the paper, we denote with p_n the largest prime factor of n. Furthermore, r_n is the smallest integer for which it holds that

$$r_n^{p_n} \equiv r_n (\text{mod } n) \quad \text{and} \quad r_n^i \neq r_n (\text{mod } n) \qquad i \in \{2,\ldots,p_n-1\}. \tag{2}$$

Although Rabung [18] only considers primitive roots for prime numbers, the general case is presented, because it will be used later on. Notice that our definition of the primitive root differs from the conventional definition for non-prime numbers. Based on r_n, a sequence S_n with $p-1$ elements can be constructed as follows:

$$S_n(i) = r_n^i (\text{mod } n) \qquad 1 \leq i \leq p_n - 1. \tag{3}$$

Finally, the elements are partitioned into k color classes such that

$$S_p(i) \in C_{i(\text{mod } k)+1} \qquad \text{and} \qquad p_n \in C_k. \tag{4}$$

Example 1. Consider the extreme certificate $W(4,3,75)$, so $p=37$, $r_{37}=2$, $k=4$.

$$\begin{aligned} S_{37} \;=\; (&2,4,\mathbf{8},\mathit{16},32,27,\mathbf{17},\mathit{34},31,25,\mathbf{13},\mathit{26},15,30,\mathbf{23},\mathit{9},18,36,\\ &\mathbf{35},\mathit{33},29,21,\mathbf{5},\mathit{10},20,3,\mathbf{6},\mathit{12},24,11,\mathbf{22},\mathit{7},14,28,\mathbf{19},\mathit{1}) \end{aligned}$$

Notice that the elements in S_{37} are shown in four different fonts. Elements with the same font have the same position in S_{37} (mod 4). First, we partition these elements according to (4), while placing element 37 in C_3 (it could have been placed in any set C_s with $s>1$). Second, we duplicate the certificate according to (1) and place element 0 in C_1. This results in

the certificate shown in Figure 3. The elements 1 to 36 are shown using the same fonts as in S_{37}.

$$
\begin{aligned}
C_1 &= \{0, \mathit{1}, 7, \mathit{9}, \mathit{10}, \mathit{12}, \mathit{16}, \mathit{26}, \mathit{33}, \mathit{34}, 38, 44, 46, 47, 49, 53, 63, 70, 71\} \\
C_2 &= \{2, 14, 15, 18, 20, 24, 29, 31, 32, 39, 51, 52, 55, 57, 61, 66, 68, 69\} \\
C_3 &= \{3, 4, 11, 21, 25, 27, 28, 30, 36, 37, 40, 41, 48, 58, 62, 64, 65, 67, 73, 74\} \\
C_4 &= \{\mathbf{5}, \mathbf{6}, \mathbf{8}, \mathbf{13}, \mathbf{17}, \mathbf{19}, \mathbf{22}, \mathbf{23}, \mathbf{35}, 42, 43, 45, 50, 54, 56, 59, 60, 72\}
\end{aligned}
$$

Figure 3: An extreme certificate $W(4, 3, 75)$ using power residue coloring.

3.2. Zipping

We proposed a method to generate certificates $W(k, l, pq(l-1)+1)$ with p prime and $q > 1$ [10]. The first step consists of computing a certificate $W(k, l, p)$ using power residue coloring. In the case q is prime, we construct a certificate $W(k, l, pq)$ by applying the *zipping rule*:

(5)

$$i \in C_s \Rightarrow iq+jp-1 (\mathrm{mod}\, pq)+1 \in C^*_{s-1+j\lceil \frac{k}{2} \rceil (\mathrm{mod}\ k)+1} \quad i \in C, j \in \{0, \ldots, q-1\}.$$

A certificate of size $pq(l-1)+1$ can be obtained by repeating the resulting $W(k, l, pq)$ a total of $l-1$ times and adding a last element.

In the case q is not prime, the zipping rule must be applied for each of the factors of q. For instance, if $q = 4$ then we apply the rule for the first factor $q_1 = 2$. The resulting certificate is extended by second factor $q_2 = 2$.

Example 2. Consider the extreme certificate $W(2, 5, 177)$ with $p = 11$, $r_p = 2$, $q_1 = 2$, $q_2 = 2$, and $k = 2$. First we compute sequence S_{11} with fonts:

$$S_{11} = (2, \mathbf{4}, 8, \mathbf{5}, 10, \mathbf{9}, 7, \mathbf{3}, 6, \mathbf{1}).$$

The elements on odd positions are placed in C_1, while the elements on even positions and 0 are placed in C_2. The first zip uses C as input and constructs a certificate C^* of double size with $q_1 = 2$. Afterwards the double certificate C^* is used as input to create a quadruple certificate C^{**} by zipping with $q_2 = 2$. The result is shown below.

In early 2007, David Mitchell and Nhan Nguyen informed us[2] that they constructed a certificate $W(3, 5, 2172)$. Recall that the old lower bound was

[2]E-mail correspondence February 2007.

$$C_1 = \{2, 6, 7, 8, 10\} \qquad C_1^* = \{4, 7, 12, 13, 14, 16, 17, 19, 20, 21\}$$
$$C_2 = \{\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{9}, 11\} \qquad C_2^* = \{1, \mathbf{2}, 3, 5, \mathbf{6}, \mathbf{8}, 9, \mathbf{10}, 11, 15, \mathbf{18}, 22\}$$

$$C_1^{**} = \{0, 3, 8, 13, 14, 15, 17, 21, 23, \mathbf{24}, 26, 27, \mathbf{28}, 29, 31, \mathbf{32}, 33, 34, 38, \mathbf{40}, 41, 42\}$$
$$C_2^{**} = \{1, 2, \mathbf{4}, 5, 6, 7, 9, 10, 11, \mathbf{12}, \mathbf{16}, 18, 19, \mathbf{20}, 22, 25, 30, 35, \mathbf{36}, 37, 39, 43, 44\}$$

965. While analyzing this certificate, many similarities were observed with the zipped certificates obtained by our generator. In fact, we were able to construct the same certificate by replacing (5) with

(6)

$$i \in C_s \Rightarrow iq + jp - 1 (\text{mod } pq) + 1 \in C^*_{s-1+j\lceil \frac{k}{q} \rceil (\text{mod } k)+1} \quad i \in C, j \in \{0, \ldots, q-1\}.$$

Notice that (5) and (6) are equivalent if $q = 2$, which was the case for all improved lower bounds using the Zipper method [10]. Hence these results are not affected by this modification. Besides improving the lower bound of $W(3, 5)$, the new generator also improved the bounds of $W(3, 6)$ (from 8886 to 11191) and $W(5, 5)$ (from 24045 to 29621). Figure 2 shows the visualizations of the new bounds of $W(3, 5)$ and $W(3, 6)$. Notice the rotation symmetry of 120° in both images.

3.3. Satisfiability

Most lower bounds to van der Waerden numbers were established using the generators described in the previous subsections. Dransfield et al. [7] proposed to translate the search for a certificate $W(k, l, n)$ as a Boolean satisfiability (SAT) problem. They showed that $W(5, 3) > 125$ using this method. Figure 4 (left) offers the visualization of the corresponding certificate. Notice that no clear pattern is observable in this image, in contrast to the visualizations of the extreme and largest known lower bound certificates (see Figure 2).

Kouril and Franco [15] improved on these results by adding clauses to the formula that are not logically implied. These additional clauses are used to guide the search. They improved the lower bound of $W(2, 6)$ from 695 to 1132. In 2008, Kouril and Paul proved that $W(2, 6) = 1132$ [16].

Existence of a certificate $W(k, l, n)$ can be naturally translated into SAT. The encoding requires kn Boolean variables $x_{i,s}$ which are true if and only if element $i \in C_s$. Two types of clauses are used. The first type ensures that each element is in *at-least-one* color class. These clauses consist only of positive literals. The second type forbids an arithmetic progression of length l in any of the color classes. These clauses consist of only negative literals. We refer to this encoding as the *minimal encoding*, which is shown in

Table 2. The formula corresponding to the existence problem of a certificate $W(k,l,n)$ is denoted by $\mathcal{F}_{k,l,n}$.

Any satisfying assignment of formula $\mathcal{F}_{k,l,n}$ would prove that $W(k,l) > n$. On the other hand, if a formula $\mathcal{F}_{k,l,n}$ is unsatisfiable then $W(k,l) \leq n$. So, these formulae can be used to determine van der Waerden numbers. To prove that $W(k,l) = n$, one needs to shows that $\mathcal{F}_{k,l,n-1}$ is satisfiable, while $\mathcal{F}_{k,l,n}$ is unsatisfiable.

Table 2: Minimal encoding of van der Waerden certificates into SAT

Variables	**Range**	**Meaning**
$x_{i,s}$	$i \in \{1,\ldots,n\}$ $s \in \{1,\ldots,k\}$	$x_{i,s} \equiv 1$ iff $i \in C_s$
Clauses	**Range**	**Meaning**
$(x_{i,1} \vee x_{i,1} \vee \cdots \vee x_{i,k})$	$i \in \{1,\ldots,n\}$	i is in at-least-one color class
$(\overline{x}_{i',s} \vee \overline{x}_{i'+d,s} \vee \ldots$ $\cdots \vee \overline{x}_{i'+d(l-1),s})$	$i' \in \{1,\ldots,n-l\}$ $d \in \{1,\ldots,\lfloor \frac{n-i'}{l-1} \rfloor\}$ $s \in \{1,\ldots,k\}$	no color class contains an arithmetic progression of length l (or larger)

Besides the clauses in Table 2, one could optionally add the binary clauses $(\overline{x}_{i,s} \vee \overline{x}_{i,s'})$ for $1 \leq i \leq n$ and $1 \leq s < s' \leq k$ [7]. These clauses express that each element i is in *at-most-one* color class. These clauses are redundant: if an element occurs in multiple color classes, then all but one of the occurrences can be removed. Notice that removing elements cannot create an arithmetic progression of length l.

Two alternative encodings have been proposed by Oliver Kullmann [17]: the *logarithmic translation* and the *weak nested translation*. Both encodings use less variables and slightly less clauses, but many more literals. In practice, the alternative encodings are faster compared to the minimal encoding. However, we will focus on the minimal encoding for the following reasons. First, after adding many auxiliary constraints (discussed in the next section) the impact on the performance is less pronounced. Second, the new techniques presented here are more naturally explained using the minimal encoding.

4. Symmetry and satisfiability

Although SAT can be used to determine van der Waerden numbers, the current state-of-the-art SAT solvers are not strong enough to find several new numbers. Therefore, some recent studies [10, 15] focus on improving lower bounds by adding constraints to the minimal encoding $\mathcal{F}_{k,l,n}$. These

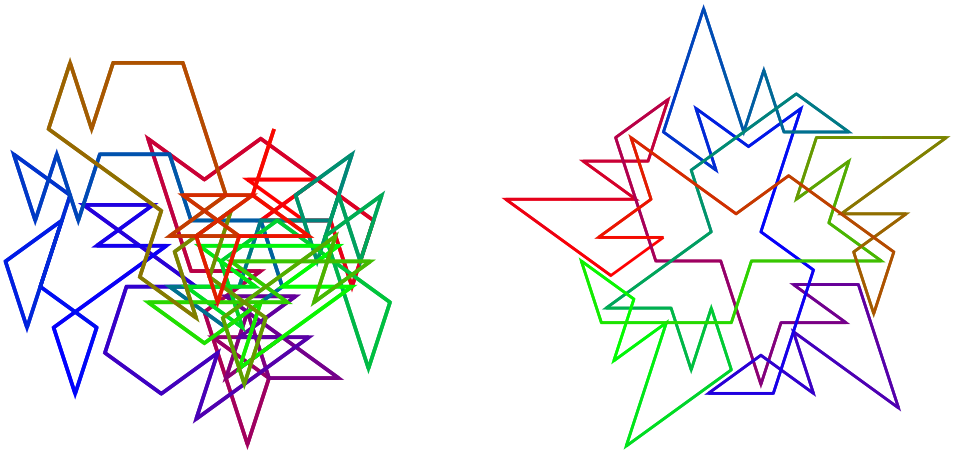

Figure 4: Visualizations of two certificates $W(5,3,n)$. Left the old lower bound $W(5,3) > 125$, right the improved lower bound $W(5,3) > 170$.

constraints force or forbid certain patterns/symmetries in the certificates. The corresponding clauses can significantly reduce the computation cost to find a certificate. Yet, since these clauses are not implied by $\mathcal{F}_{k,l,n}$, they can only be used to establish lower bounds for the van der Waerden numbers.

First, we demonstrate how to exploit the observed rotation symmetry in the visualizations to establish improved lower bounds for $W(5,3)$ and $W(6,3)$ in Section 4.1. In Section 4.2, we analyze these initial results. This inspired us to develop two new techniques: *pre-partitioning* (Section 4.3) and *internal symmetries* (Section 4.4).

4.1. Repetition, reflection, and rotation

A common approach to boost performance on hard combinatorial problem adds auxiliary constraints to guide (also known as streamline) the search [8, 15, 10, 12]. We present three types of auxiliary constraints to improve the performance of finding van der Waerden certificates.

The first type originates from the observation that most extreme certificates and the best known lower bounds of $W(k,l)$ show a repetition of $l-1$ times the same pattern. We refer to this symmetry as the *repetition symmetry*. It can be forced by adding the constraints $x_{i,s} \leftrightarrow x_{i+m,s}$ with $i \in \{1,\ldots,m(l-2)\}$ and $s \in \{1,\ldots,k\}$. Addition of these constraints has been studied in the past [10]. However, the addition was not sufficient to improve some lower bounds.

Recall the extreme certificate of $W(4,3,75)$ in Figure 3 that was produced using power residue coloring. Notice that if element i occurs in color class C_s then element $m-i$ occurs in color class $C_{s+1(\mathrm{mod}\ 4)+1}$. By swapping the color classes C_3 and C_4, this relation becomes: $i \in C_s$ implies $m-i$ occurs in color class C_{k+1-s}. This pattern results in the reflection symmetry in the visualization of certificates in Figure 2. We therefore refer to this pattern and the second type of auxiliary constraints as the reflection symmetry.

The third type was inspired by the visualizations of certificates. Recall the improved lower bounds of $W(3,5)$, $W(3,6)$, and $W(5,5)$ that were found by the new generator. Figure 2 shows certificates $W(3,5,2173)$ and $W(3,6,11191)$. All these visualizations show a clear rotation by $\frac{360^\circ}{k}$ — a symmetry that was not observed before. It appeared that this rotation was the result of zipping with $q=k$. All these improved bounds are of the form $W(k,l,pk(l-1)+1)$ with p prime. This *rotation symmetry* can be forced by adding the constraints $x_{i,s} \leftrightarrow x_{i+p,s+1(\mathrm{mod}\ k)}$ for $i \in \{1,\ldots,pk\}$ and $s \in \{1,\ldots,k\}$. Table 3 shows both constraints as clauses.

Table 3: Encoding of repetition, reflection, rotation and root symmetries

Clauses	Range	Meaning
$(\overline{x}_{i,s} \vee x_{i+m,s}) \wedge (x_{i,s} \vee \overline{x}_{i+m,s})$	$i \in \{1,\ldots,ml-m\}$, $s \in \{1,\ldots,k\}$	repetition symmetry $\sigma^{\rightarrow\rightarrow}_{k,m}$
$(\overline{x}_{i,s} \vee x_{m-i,k+1-s}) \wedge (x_{i,s} \vee \overline{x}_{m-i,k+1-s})$	$i \in \{1,\ldots,\frac{m}{2}\}$, $s \in \{1,\ldots,k\}$	reflection symmetry $\sigma^{\rightleftarrows}_{k,m}$
$(\overline{x}_{i,s} \vee x_{i+p_m,s(\mathrm{mod}\ k)+1}) \wedge (x_{i,s} \vee \overline{x}_{i+p_m,s(\mathrm{mod}\ k)+1})$	$i \in \{1,\ldots,m-p_m\}$, $s \in \{1,\ldots,k\}$	rotation symmetry $\sigma^{\circlearrowleft}_{k,m}$
$(\overline{x}_{r_m^i(\mathrm{mod}\ m),s} \vee x_{r_m^{i+a}(\mathrm{mod}\ m),s}) \wedge (x_{r_m^i(\mathrm{mod}\ m),s} \vee \overline{x}_{r_m^{i+a}(\mathrm{mod}\ m),s})$	$i \in \{1,\ldots,p_m\}$, $s \in \{1,\ldots,k\}$	root symmetry $\sigma^{\mathrm{root}}_{a,k,m}$

4.2. Initial results and analysis

Our first goal was to improve $W(5,3) > 125$ reported by Dransfield et al. [7]. We generated the formulas $\mathcal{F}_{5,3,n}$ with $125 \le n \le 200$ and added the symmetries $\sigma^{\rightarrow\rightarrow}_{k,m}$, $\sigma^{\rightleftarrows}_{k,m}$ and $\sigma^{\circlearrowleft}_{k,m}$. We found no solution for any of the formulas that had all symmetries enforced. As a consequence we experimented with enforcing some, but not all observed symmetries. We found a solution for $\mathcal{F}_{5,3,170}$ by discarding the forced symmetry $\sigma^{\rightleftarrows}_{k,m}$, thereby improving the lower bound significantly. The corresponding certificate of the solution is shown in Figure 6 and visualized in Figure 4 (right).

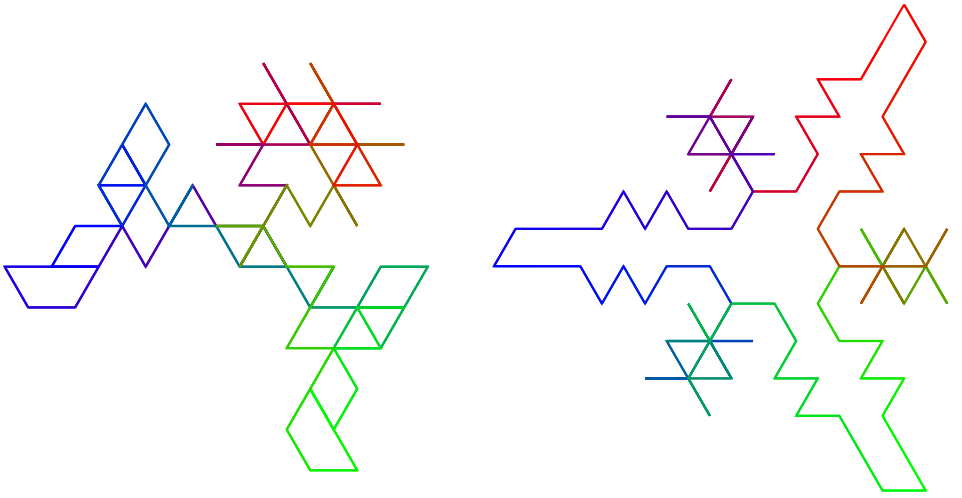

Figure 5: Visualizations of two certificates $W(6,3,n)$. Left the old lower bound $W(6,3) > 207$, right the improved lower bound $W(6,3) > 223$.

Encouraged by the improvement for $W(5,3)$, we did similar experiments for $W(6,3)$. This lower bound was 207 and is visualized in Figure 5 (left). For none of the formulas $\mathcal{F}_{6,3,n}$ with $206 \leq n \leq 300$, did we found a solution by forcing the symmetries as presented in Table 3. However, after slightly modifying the rotation symmetry to $(\overline{x}_{i,s} \vee x_{i+p_m,s+1(\bmod\ k)+1}) \wedge (x_{i,s} \vee \overline{x}_{i+p_m,s+1(\bmod\ k)+1})$, a solution of $\mathcal{F}_{6,3,233}$ was found. The corresponding certificate of this solution is shown in Figure 7 and visualized in Figure 5 (right).

Why are the generator techniques not able to find these certificates, which contain the observed (and forced) symmetries? Can we find new symmetries in the certificates of the improved lower bounds?

Consider how using the zipping method would construct a certificate $W(k,l,pq(l-1))$ or $W(k,l,pq(l-1)+1)$. First S_p would be computed and the sets C_j would be populated with the elements $q \cdot S_p(i)$ if $j \equiv i(\bmod k)+1$ with $i \in \{1,\ldots,p\}$ and $j \in \{1,\ldots,q\}$. Concretely for $W(5,3,170)$, thus $p = 17$ and $q = 5$, this would mean $5 \cdot S_{17}(i) \in C_j$ if $j \equiv i(\bmod 5) + 1$. There are no such certificates. However, there exists a slightly different pattern in the certificate $W(5,3,170)$ computed by enforced symmetries: $5 \cdot S_{17}(i) \in C_j$ if $j \equiv i(\bmod 4) + 1$. To observe this pattern, consider S_{17} in which each element uses a font based on its position (mod 4).

$$S_{17} \quad = \quad (3, 9, \mathbf{10}, \mathit{13}, 5, 15, \mathbf{11}, \mathit{16}, 14, 8, \mathbf{7}, \mathit{4}, 12, 2, \mathbf{6}, \mathit{1})$$

Figure 6 shows the first $m = 85$ elements of the certificate $W(5, 3, 170)$ found by enforcing symmetries. Notice that if two elements x and y have the same font in S_{17} shown above, then $5 \cdot x$ and $5 \cdot y$ occur in the same set of the certificate.

$$
\begin{aligned}
C_1 &= \{14, \mathtt{15}, 23, \mathtt{25}, 28, 29, 39, 47, 51, 52, 54, 58, \mathtt{60}, 67, \mathtt{70}, 72, 78\} \\
C_2 &= \{2, 4, \mathtt{10}, 31, 32, \mathtt{40}, 42, \mathtt{45}, 46, 56, 64, 68, 69, 71, \mathtt{75}, 77, 84\} \\
C_3 &= \{6, 8, 11, 12, 22, \mathbf{30}, 34, \mathbf{35}, 37, 41, 43, \mathbf{50}, 53, \mathbf{55}, 61, 82, 83\} \\
C_4 &= \{\mathit{5}, 13, 17, 18, \mathit{20}, 24, 26, 33, 36, 38, 44, \mathit{65}, 66, 74, 76, 79, \mathit{80}\} \\
C_5 &= \{1, 3, 7, 9, 16, 19, 21, 27, 48, 49, 57, 59, 62, 63, 73, 81, 85\}
\end{aligned}
$$

Figure 6: The first $m = 85$ elements of a certificate $W(5, 3, 170)$ which was obtained by enforcing repetition and rotation symmetries. This certificate is visualized in Figure 4 (right).

The found certificate $W(6, 3, 223)$ differs in a similar way. Here we have $p = 37$ and $q = 3$, so we would expect $3 \cdot S_{37}(i) \in C_j$ if $j \equiv i \pmod 6 + 1$. Again no certificates of this type exist. However, in the found certificate $3 \cdot S_{37}(i) \in C_j$ if $j \equiv i \pmod 4 + 1$ can be observed. Recall S_{37} for which elements that occur in the same position (mod 4) have the same font:

$$
\begin{aligned}
S_{37} = (&2, 4, \mathbf{8}, \mathit{16}, \mathtt{32}, 27, \mathbf{17}, \mathit{34}, 31, 25, \mathbf{13}, \mathit{26}, \mathtt{15}, 30, \mathbf{23}, \mathit{9}, 18, 36, \\
&\mathbf{35}, \mathit{33}, 29, 21, \mathbf{5}, \mathit{10}, 20, 3, \mathbf{6}, \mathit{12}, 24, 11, \mathbf{22}, \mathit{7}, 14, 28, \mathbf{19}, \mathit{1})
\end{aligned}
$$

Notice that if two elements x and y have the same font in S_{37} shown above, then $3 \cdot x$ and $3 \cdot y$ occur in the same set of the certificate.

$$
\begin{aligned}
C_1 &= \{4, \mathtt{6}, 25, 28, 40, \mathtt{42}, \mathtt{45}, \mathtt{54}, 58, \mathtt{60}, 64, 67, \mathtt{72}, 73, 85, \mathtt{87}, \mathtt{93}, \mathtt{96}, \mathtt{111}\} \\
C_2 &= \{\mathtt{9}, \mathtt{12}, 31, \mathtt{33}, 52, 55, 61, \mathtt{63}, \mathtt{75}, 76, \mathtt{81}, \mathtt{84}, 88, \mathtt{90}, 94, 103, 106, \mathtt{108}\} \\
C_3 &= \{\mathbf{15}, \mathbf{18}, \mathbf{24}, 26, 38, \mathbf{39}, 44, 47, \mathbf{51}, 53, \mathbf{57}, \mathbf{66}, \mathbf{69}, 71, 83, 86, \mathbf{105}, 107\} \\
C_4 &= \{\mathit{3}, 5, 8, 17, \mathit{21}, 23, \mathit{27}, \mathit{30}, 35, \mathit{36}, \mathit{48}, 50, 56, 59, \mathit{78}, 80, \mathit{99}, \mathit{102}\} \\
C_5 &= \{1, 2, 7, 10, 14, 16, 20, 29, 32, 34, 37, 46, 49, 68, 70, 89, 92, 98, 100\} \\
C_6 &= \{11, 13, 19, 22, 41, 43, 62, 65, 74, 77, 79, 82, 91, 95, 97, 101, 104, 109, 110\}
\end{aligned}
$$

Figure 7: The first $m = 111$ elements of a certificate $W(6, 3, 223)$ which was obtained by enforcing repetition and reflection symmetries. This certificate is visualized in Figure 5 (right).

4.3. Pre-partitioning

We were able to improve the lower bounds of $W(5,3)$ and $W(6,3)$ by restricting the search to certificates with observed symmetries. However, this method appeared not suitable to improve lower bounds of $W(k,3)$ for $k > 6$. This section introduces a technique —that emerged from analyzing improved bounds— which works for larger values of k as well. A careful look at the certificate $W(5,3,85)$ reveals that the following relationship holds for all elementst: $i \in C_j \Rightarrow 81 \cdot i \pmod{85} \in C_j$. A similar pattern can be found in $W(6,3,111)$. In this certificate, the following relationship holds for all elements: $i \in C_j \Rightarrow 16i \pmod{111} \in C_j$. Recall that $p_{85} = 17$ and $p_{111} = 37$. Moreover, the multiplication factor 81 (for $W(5,3,85)$) and 16 (for $W(6,3,111)$) are related to the primitive root: $r_{85}^4 = 3^4 = 81$ and $r_{111}^4 = 2^4 = 16$.

These observations are the motivation for the concept *pre-partitioning*. A pre-partition of $\{1,\ldots,m\}$ using parameter t, creates a partition, denoted by $PP(m,r^t)$, that groups elements x and y if and only if there exists an i such that $y = x \cdot r^{i \cdot t} \pmod{m}$. For example, with $m = 44$ (thus $r = 3$) and $t = 2$, the elements 1 and 5 are grouped, because $5 = 1 \cdot 3^{4 \cdot 2} \pmod{44}$. The full pre-partition using $m = 44$ and $t = 2$ is shown below.

$$\begin{aligned} PP(44,3^2) = \{&\{1,5,9,25,37\},\{2,6,10,18,30\},\{3,15,23,27,31\},\\ &\{4,12,16,20,36\},\{7,19,35,39,43\},\{8,24,28,32,40\},\{11\},\\ &\{13,17,21,29,41\},\{14,26,34,38,42\},\{22\},\{33\},\{44\}\} \end{aligned}$$

Using this pre-partition, it is possible to construct the first cycle of an extreme certificate of $W(2,5)$:

C_1={{1,5,9,25,37},{7,19,35,39,43},{8,24,28,32,40},{13,17,21,29,41},{22},{33}}
C_2={{2,6,10,18,30},{3,15,23,27,31},{4,12,16,20,36},{11},{14,26,34,38,42},{44}}.

Notice that the search space to find such a certificate is much smaller: instead of requiring 44 Boolean variables, we only need 12 Boolean variables (i.e., one for each group of elements). In a similar way, certificates $W(5,3,170)$ and $W(6,3,223)$ can be found using pre-partitions $PP(85,3^4)$ (Figure 8) and $PP(111,2^4)$ (Figure 9), respectively.

4.4. Internal symmetry

A solution symmetry maps any solution (certificate) onto another solution. The problem of whether there exists a certificate $W(k,l,n)$ has two solution

$$
\begin{aligned}
C_1 &= \{\{1,16,21,81\},\{3,48,63,73\},\{7,27,57,62\},\{9,19,49,59\},\{85\}\}\\
C_2 &= \{\{5,20,65,80\},\{13,18,33,38\},\{17\},\{24,44,74,79\},\{26,36,66,76\}\}\\
C_3 &= \{\{6,11,41,61\},\{8,43,53,83\},\{12,22,37,82\},\{30,35,50,55\},\{34\}\}\\
C_4 &= \{\{14,29,39,54\},\{15,25,60,70\},\{23,28,58,78\},\{47,52,67,72\},\{51\}\}\\
C_5 &= \{\{2,32,42,77\},\{4,64,69,84\},\{10,40,45,75\},\{31,46,56,71\},\{68\}\}
\end{aligned}
$$

Figure 8: The first cycle of a certificate for $W(5,3)$ based on $PP(85,3^4)$.

$$
\begin{aligned}
C_1 &= \{\{1,7,10,16,34,46,49,70,100\},\{2,14,20,29,32,68,89,92,98\},\{37\}\}\\
C_2 &= \{\{3,21,27,30,36,48,78,99,102\},\{5,8,17,23,35,50,56,59,80\}\}\\
C_3 &= \{\{4,25,28,40,58,64,67,73,85\},\{6,42,45,54,60,72,87,93,96\},\{111\}\}\\
C_4 &= \{\{9,12,33,63,75,81,84,90,108\},\{31,52,55,61,76,88,94,103,106\}\}\\
C_5 &= \{\{11,41,62,65,77,95,101,104,110\},\{13,19,22,43,79,82,91,97,109\},\{74\}\}\\
C_6 &= \{\{15,18,24,39,51,57,66,69,105\},\{26,38,44,47,53,71,83,86,107\}\}
\end{aligned}
$$

Figure 9: The first cycle of a certificate for $W(6,3)$ based on $PP(111,2^4)$.

symmetries. First, for any certificate with no arithmetic progression length l, it holds that any permutation of the color classes results in a certificate which has also no arithmetic progression of length l. We refer to this solution symmetry as σ_{color} which represents a possible permutation of the color classes. Second, for any certificate $W(k,l,n)$ we can construct another certificate by replacing each element i by $n+1-i$. We refer to this solution symmetry as σ_{invert} because all elements are inverted within the domain $\{1,\ldots,n\}$.

In the case a problem contains solution symmetries it is good practice to break them: add constraints that focus the search on a particular solution of a symmetry group. These constraints, called *symmetry breaking predicates*, reduce that search space and thereby reduce that cost to solve a problem (especially if the problem has no solutions). An example of symmetry breaking predicates for σ_{color} in $\mathcal{F}_{k,l,n}$ are clauses the force the first element to be in the first color class ($x_{1,1}$), the second element to be in the first two color classes ($x_{2,1} \vee x_{2,2}$) till the $k-1$ elements to be in the first $k-1$ color classes ($x_{k-1,1} \vee \cdots \vee x_{k-1,k-1}$). Although these clauses are useful, in practice the reduction of the search space is too small to find larger lower bounds.

An alternative technique that exploits symmetries is the concept of *internal symmetries* [12]. An internal symmetry σ_{internal} is a (non-trivial) map-

18	20	24	26	33	36	38	44	65	66	74	76	79	80	5	13	17
22	30	34	35	37	41	43	50	53	55	61	82	83	6	8	11	12
23	25	28	29	39	47	51	52	54	58	60	67	70	72	78	14	15
31	32	40	42	45	46	56	64	68	69	71	75	77	84	2	4	10
19	21	27	48	49	57	59	62	63	73	81	85	1	3	7	9	16

$\sigma_{+p_m} \Uparrow$ $\Downarrow \sigma_{\text{color}}$

1	3	7	9	16	19	21	27	48	49	57	59	62	63	73	81	85
5	13	17	18	20	24	26	33	36	38	44	65	66	74	76	79	80
6	8	11	12	22	30	34	35	37	41	43	50	53	55	61	82	83
14	15	23	25	28	29	39	47	51	52	54	58	60	67	70	72	78
2	4	10	31	32	40	42	45	46	56	64	68	69	71	75	77	84

$\sigma_{\times r_m} \Downarrow$ $\Uparrow \sigma_{\text{color}}$

3	9	21	27	48	57	63	81	59	62	1	7	16	19	49	73	85
15	39	51	54	60	72	78	14	23	29	47	25	28	52	58	67	70
18	24	33	36	66	5	17	20	26	38	44	65	74	80	13	76	79
42	45	69	75	84	2	32	56	68	71	77	4	10	31	40	46	64
6	12	30	8	11	35	41	50	53	83	22	34	37	43	55	61	82

Figure 10: Diagram showing that certificate $W(5, 3, 170)$ in Figure 6 ($m = 85$) has internal symmetries $\sigma_{+p_m} \circ \text{color}$ and $\sigma_{\times r} \circ \text{color}$. Applying σ_{+p_m} ($p_{85} = 17$), the certificate in the middle is transformed to the top certificate. Sorting the elements in the top certificate and permuting the rows (σ_{color}) maps it back. A similar scheme is observed using $\sigma_{\times r_m}$ ($r_{85} = 3$).

ping of a solution (certificate) onto itself. Let an observed symmetry σ_{observed} be a mapping of a (specific) solution onto another solution. Internal symmetries can frequently be decomposed into an observed symmetry and a solution symmetry: $\sigma_{\text{internal}} = \sigma_{\text{observed}} \circ \sigma_{\text{solution}}$. Figure 10 shows two examples of internal symmetries of the certificate improving the lower bound of $W(5, 3)$.

We experimented with three types of internal symmetries. The first is a generalization of the rotation symmetry. Recall that there exists a certificate $W(5, 3, 170)$ containing the symmetry $x_{i,s} \leftrightarrow x_{i=p_m, s(\text{mod } k)+1}$, while there exists a certificate $W(6, 3, 111)$ containing the symmetry $x_{i,s} \leftrightarrow x_{i=p_m, s+1(\text{mod } k)+1}$ for $i \in \{1, \ldots, m - p_m\}$ and $s \in \{1, \ldots, k\}$. We refer to this internal symmetry as $\sigma_{+p_m} \circ \text{color}$. The clauses that enforce this symmetry are shown in Table 4. The clauses $(\overline{y}_{+p_m,i,j} \vee \overline{y}_{+p_m,i,j'})$ enforce that for each i at-most-one $y_{+p_m,i,j}$ can be assigned to true. During the experiments

Table 4: Additional constraints that force internal symmetry $\sigma_{+p_m} \circ \text{color}$

Clauses	Range
$(\overline{x}_{i,h} \vee \overline{x}_{j,h+p_m \pmod n} \vee y_{+p_m,i,j})$	$i,j \in \{1,\ldots,l\}, h \in \{1,\ldots,n\}, h \not\equiv 0 \pmod{p_m}$
$(\overline{y}_{+p_m,i,j} \vee \overline{y}_{+p_m,i,j'})$	$i,j \in \{1,\ldots,l\}$, $j' \in \{j+1,\ldots,l\}$
$(\overline{y}_{+p_m,i,j} \vee \overline{y}_{+p_m,i',j})$	$i,j \in \{1,\ldots,l\}$, $i' \in \{i+1,\ldots,l\}$

Table 5: Additional constraints that force internal symmetry $\sigma_{\times r} \circ \text{color}$

Clauses	Range
$(\overline{x}_{i,h} \vee \overline{x}_{j,h\times r_m \pmod n} \vee y_{\times r_m,i,j})$	$i,j \in \{1,\ldots,l\}, h \in \{1,\ldots,n\}, h \not\equiv 0 \pmod{p_m}$
$(\overline{y}_{\times r_m,i,j} \vee \overline{y}_{\times r_m,i,j'})$	$i,j \in \{1,\ldots,l\}$, $j' \in \{j+1,\ldots,l\}$
$(\overline{y}_{\times r_m,i,j} \vee \overline{y}_{\times r_m,i',j})$	$i,j \in \{1,\ldots,l\}$, $i' \in \{i+1,\ldots,l\}$

Table 6: Additional constraints that force internal symmetry $\sigma_{\text{inv}} \circ \text{color}$

Clauses	Range
$(\overline{x}_{i,h} \vee \overline{x}_{j,n-h} \vee y_{\text{inv},i,j})$	$i,j \in \{1,\ldots,l\}, i \leq j, h \in \{1,\ldots,n\}, h \not\equiv 0 \pmod{p_m}$
$(\overline{y}_{\text{inv},i,j} \vee \overline{y}_{\text{inv},i,j'})$	$i,j \in \{1,\ldots,l\}$, $j' \in \{j+1,\ldots,l\}$
$(\overline{y}_{\text{inv},i,j} \vee \overline{y}_{\text{inv},i',j})$	$i,j \in \{1,\ldots,l\}$, $i' \in \{i+1,\ldots,l\}$

we noticed that his constraint was sometimes too strong. The constraint can be weakened by enforcing that for each i at-most-two $y_{+p_m,i,j}$ can be assigned to true. In case the weakened version is used, we refer to it as $\sigma_{+\text{p}_\text{m}} \circ \text{color}^*$.

The second internal symmetry is closely related to pre-partitioning. Pre-partitioning forces elements to be in the same set based on the primitive root r_m. Internal $\sigma_{\times r_m} \circ \text{color}$ generalizes this pattern by relating the elements occurring in different sets based on r_m. Table 5 shows the clauses expressing this internal symmetry.

The third internal symmetry is a generalization of the reflection symmetry $\sigma^{\rightleftarrows}_{k,m}$. Recall that for most van der Waerden numbers there exists an extreme or largest known certificate with the reflection symmetry. However, $\sigma^{\rightleftarrows}_{k,m}$ does not cover a slightly different pattern that can be observed in some extreme certificates of $W(3,3,26)$. These certificates contain the symmetry $x_{i,s} \leftrightarrow x_{n+1-i,k+1-s}$ for $i \in \{1,\ldots,n\}$ and $s \in \{1,\ldots,k\}$. In this variant, each color class reflects onto itself. The internal symmetry $\sigma_{\text{inv}} \circ \text{color}$ covers both variants: each color class either reflects onto itself or onto another color class. The clauses representing $\sigma_{\text{inv}} \circ \text{color}$ are shown in Table 6.

Notice that the number of additional clauses for each of these internal symmetries is $\mathcal{O}(k^2 n)$, hence much smaller than the number of original clauses.

5. Results

This section offers improved lower bounds of van der Waerden numbers that were obtained using the pre-partitioning and internal symmetry techniques discussed above. First, we describe the experimental setup. Second, we present improved lower bounds of $W(7,3)$, $W(8,3)$, and $W(9,3)$ together with visualizations of the new bounds. At the end, we present some results for improved lower bounds of some other van der Waerden numbers.

5.1. Experimental setup

The experiments were performed on the Lonestar 5 cluster of the Texas Advanced Computing Center (TACC). Each computing node has a Xeon E5-2690 v3 with two 12 core chips. We ran 48 settings in parallel for each $W(k,l)$ lower bound computation with a five minute timeout for each SAT call. The SAT solver glucose 3.0 [3] was used during all experiments. We implemented a tool that generates the SAT formula that encodes the existence of a lower bound for a given $W(k,l)$ and given enforced patterns. The tool is available at https://github.com/marijnheule/vdWaerden as well as the certificates discussed in this paper.

Over the last couple of years, we experimented using a large variety of enforced patterns to improve lower bounds. The useful patterns were generalized into internal symmetries and pre-partitioning. In the following subsections, we briefly describe how we improved the lower bounds for $W(7,3)$, $W(8,3)$, and $W(9,3)$.

5.2. Tree Frog

Although $W(7,3)$ is the smallest van der Waerden number considered in this section, improving its lower bound substantially was hard compared to the other results. With existing methods we constructed a certificate $W(7,3,267)$, but many attempts to find a larger one failed. The rotation symmetry which was helpful to improve the lower bounds of $W(5,3)$ and $W(6,3)$, was not useful for $W(7,3)$. After many days of search, we found a certificate $W(7,3,342)$ which is shown in Figure 11 and visualized in Figure 12. We named the certificate *Tree Frog* due to its resemblance of the visualization with the amphibian in both shape and color (blue-green body and red feet).

Figure 11 also shows how to obtain the certificate using the presented methods: use pre-partition $PP(171,2^6)$ and enforcing a reflection symmetry. Later on, after developing the concept of internal symmetries, the same

$$
\begin{aligned}
C_1 &= \{\{2,128,155\},\{3,21,147\},\{6,42,123\},\{15,51,105\},\\
&\quad \{45,144,153\},\{50,113,122\},\{68,77,140\},\{74,92,119\},\{76\}\}\\
C_2 &= \{\{8,107,170\},\{9,63,99\},\{12,75,84\},\{26,125,134\},\\
&\quad \{29,110,146\},\{30,39,102\},\{33,60,78\},\{47,101,137\},\{133\}\}\\
C_3 &= \{\{4,85,139\},\{11,20,83\},\{13,67,148\},\{14,41,59\},\{19\},\\
&\quad \{44,80,161\},\{55,73,100\},\{56,65,164\},\{109,136,154\},\{171\}\}\\
C_4 &= \{\{5,131,149\},\{22,40,166\},\{28,82,118\},\{36,54,81\},\\
&\quad \{53,89,143\},\{57\},\{90,117,135\},\{114\}\}\\
C_5 &= \{\{0\},\{7,106,115\},\{10,91,127\},\{17,35,62\},\{23,104,158\},\\
&\quad \{32,86,167\},\{71,98,116\},\{88,151,160\},\{112,130,157\},\{152\}\}\\
C_6 &= \{\{1,64,163\},\{25,61,142\},\{34,70,124\},\{37,46,145\},\{38\},\\
&\quad \{69,132,141\},\{72,108,162\},\{87,96,159\},\{93,111,138\}\}\\
C_7 &= \{\{16,43,169\},\{18,27,126\},\{24,150,168\},\{31,94,103\},\\
&\quad \{48,129,165\},\{49,58,121\},\{52,79,97\},\{66,120,156\},\{95\}\}
\end{aligned}
$$

Figure 11: The first cycle of a reflective certificate $W(7,3,342)$ with elements grouped based on $PP(171,2^6)$.

certificate was obtained by enforcing $\sigma_{r^3 \circ \text{color}}$ with $r = 2$. Notice that the certificate has no rotation symmetry in contrast to the other certificates.

5.3. Dying Tulip

Pre-partitioning was very effective to establish a decent lower bound for $W(8,3)$. Using $PP(255,7^4)$, we found a certificate $W(8,3,511)$ which is shown in Figure 13. The visualization of this certificate, shown in Figure 14, we named *Dying Tulip* because of the similarity of the image and the flower that bends and opens the leaves while dying.

Notice that there is a large resemblance between the construction of the largest found certificates for $W(6,3)$ and $W(8,3)$. The former is constructed by 3-zipping the largest possible certificate for $W(4,3)$, while the latter can be obtained by 3-zipping the largest known certificate for $W(5,3)$. After zipping, both certificates can be extended with one additional element.

5.4. Spiky Rose

Improving the lower bound of $W(9,3)$ was relatively easy compared to the other two improvements ($W(7,3)$ and $W(8,3)$). By analyzing the improved

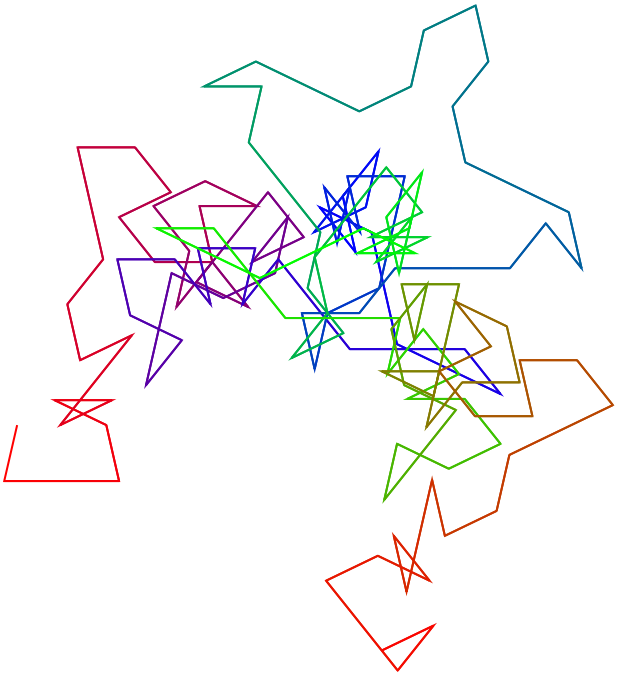

Figure 12: Visualization of "Tree Frog" showing that $W(7,3) > 342$.

bound of $W(6,3)$, several patterns could be observed, which can be described as internal symmetries σ_{+m}, $\sigma_{+p \circ \text{color}}$, and $\sigma_{\times r^2 \circ \text{color}}$. We first created a generator that searched for certificates with these patterns. That generator found a certificate $W(9,3,763)$, but was unable to improve the bounds for $W(7,3)$ and $W(8,3)$. The same certificate was later produced via SAT using pre-partition $PP(381, 7^6)$. Figure 15 shows that certificate and Figure 16 shows a visualization which we named "Spiky Rose". Notice that this certificate shows that $W(9,3)$ is substantially larger than $9^3 = 726$. In contrast, the improved bounds for $W(7,3)$ and $W(8,3)$ are equal to 7^3 and 8^3, respectively.

5.5. Overview

During our initial experiments to compute lower bounds for $W(k,3)$ with $k \geq 5$, we observed that all improved lower bounds had the σ_{m+} internal symmetry and could be computed using a pre-partition $PP(m, r_m^t)$ for some $t \in \{2, \ldots, 25\}$. We decided to enforce both patterns during all runs. Addi-

C_1={{11,41,146,176},{17},{27,57,147,177},{30,120,135,225},{36,66,111,246},
{39,54,99,114},{89,149,239,254},{110,155,185,230},{158,173,218,233},{153}}
C_2={{1,16,106,166},{8, 53, 83, 128},{28,58,163,193},{34},{44,74,164,194},
{47,137,152,242},{56,71,116,131},{127,172,202,247},{175,190,235,250},{170}}
C_3={{4,64,154,169},{9,144,189,219},{12,192,207,252},{18,33,123,183},{51},
{25,70,100,145},{45,75,180,210},{61,91,181,211},{73,88,133,148},{187}}
C_4={{14,29,209,224},{21,81,171,186},{26,161,206,236},{35,50,140,200},{68},
{42,87,117,162},{62,92,197,227},{78,108,198,228},{90,105,150,165},{204}}
C_5={{31,46,226,241},{38,98,188,203},{43,178,223,253},{52,67,157,217},{85},
{59,104,134,179},{79,109,214,244},{95,125,215,245},{107,122,167,182},{221}}
C_6={{3,48,63,243},{6,96,126,231},{7,112,142,232},{15,60,195,240},{102},
{55,115,205,220},{69,84,174,234},{76,121,151,196},{124,139,184,199},{238}}
C_7={{2,32,77,212},{5,20,65,80},{23,113,143,248},{24,129,159,249},{119},
{24,129,159,249},{72,132,222,237},{93,138,168,213},{141,156,201,216},{255}}
C_8={{10,40,130,160},{13,103,118,208},{19,49,94,229},{22,37,82,97},{136}}

Figure 13: The first cycle of $W(8, 3, 511)$ which was obtained using $PP(255, 7^4)$.

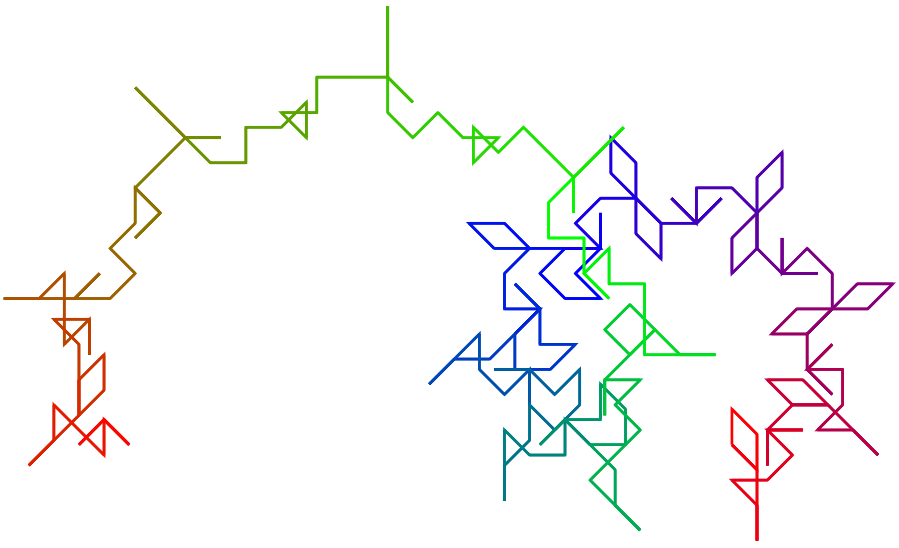

Figure 14: Visualization of "Dying Tulip" showing that $W(8, 3) > 511$.

tionally, we used internal symmetry $\sigma_{p+\circ \text{color}^*}$ or $\sigma_{\text{m}-\circ \text{color}}$. The algorithm below shows the pseudo-code of our final experiments.

Algorithm 1 SearchLowerBound (k, l)

for $\frac{LowerBound(k,l)}{l-1} \leq m \leq \frac{LowerBound(k+1,l)}{l-1}$ **do**
  $n := m(l-1)$
  for $t \in \{2, \ldots, 25\}$ **do**
    if SOLVE $(F_{k,l,n} \cup \sigma_{m+} \cup PP(m, r_m^t) \cup \sigma_{p+\circ \text{color}^*}) = \textit{satisfiable}$ **then**
      LowerBound $(k, l) := n$
    end if
    if SOLVE $(F_{k,l,n} \cup \sigma_{m+} \cup PP(m, r_m^t) \cup \sigma_{\text{m}-\circ \text{color}}) = \textit{satisfiable}$ **then**
      LowerBound $(k, l) := n$
    end if
  end for
end for

The results of those experiments are shown in Table 7 (pre-partition) and Table 8 (internal symmetries). The bold numbers in Table 7 show the improved lower bounds. Notice that for all lower bounds of $W(k, l)$ that have not been improved it holds that there exists largest known certificates that can be obtained with a pre-partition $PP(k, r_m^k)$. This property does not hold for any of the improved lower bounds. This is probably the most clear difference between the presented SAT approach and existing methods such as power residue coloring [18] and zipping [10, 19]. The question arises whether these methods can be generalized such that they can produce these improved lower bounds.

6. Open problems

Using the presented methods, we were able to improve several lower bounds of van der Waerden numbers. These methods can also be used to produce extreme certificates of known van der Waerden numbers. This section offers some open problems that are inspired by the presented results.

6.1. Avoiding patterns yields patterns

One of the main open questions is whether there exists extreme certificates with internal symmetries for all van der Waerden numbers. In other words, avoiding the pattern of arithmetic progressions yields other patterns which can be expressed using internal symmetries. For all known van der Waerden numbers there exists an extreme certificate with an internal symmetry. That

$$
\begin{aligned}
C_1 &= \{\{0\}, \{11, 17, 41, 44, 68, 74, 98, 149, 161, 164, 176, 209, 215, 263,\\
&\quad 272, 275, 290, 296, 323, 326, 338\}, \{13, 31, 52, 70, 79, 103, 115, 121,\\
&\quad 124, 142, 157, 187, 208, 226, 247, 280, 289, 316, 325, 358, 367\}\}\\
C_2 &= \{\{10, 40, 154, 160, 178, 181, 190, 193, 229, 235, 238, 250, 253, 259,\\
&\quad 274, 331, 334, 343, 349, 373, 379\}, \{87, 117, 129, 135, 159, 165, 174,\\
&\quad 177, 234, 249, 255, 258, 270, 273, 279, 315, 318, 327, 330, 348, 354\}\}\\
C_3 &= \{\{3, 6, 12, 24, 48, 57, 75, 96, 114, 141, 150, 183, 192, 219, 228, 261,\\
&\quad 282, 300, 321, 351, 366\}, \{29, 53, 59, 83, 86, 110, 116, 170, 182,\\
&\quad 185, 212, 218, 233, 236, 245, 293, 299, 332, 344, 347, 359\}, \{127\}\}\\
C_4 &= \{\{9, 18, 21, 36, 42, 69, 72, 84, 138, 144, 168, 171, 195, 201, 225, 276,\\
&\quad 288, 291, 303, 336, 342\}, \{26, 35, 62, 71, 104, 113, 140, 158, 179,\\
&\quad 197, 206, 230, 242, 248, 251, 269, 284, 314, 335, 353, 374\}\}\\
C_5 &= \{\{1, 4, 16, 19, 25, 61, 64, 73, 76, 94, 100, 214, 244, 256, 262, 286,\\
&\quad 292, 301, 304, 361, 376\}, \{5, 20, 77, 80, 89, 95, 119, 125, 137, 167,\\
&\quad 281, 287, 305, 308, 317, 320, 356, 362, 365, 377, 380\}\}\\
C_6 &= \{\{7, 28, 46, 67, 97, 112, 130, 133, 139, 151, 175, 184, 202, 223, 241,\\
&\quad 268, 277, 310, 319, 346, 355\}, \{39, 45, 78, 90, 93, 105, 156, 180, 186,\\
&\quad 210, 213, 237, 243, 297, 309, 312, 339, 345, 360, 363, 372\}, \{254\}\}\\
C_7 &= \{\{15, 30, 60, 81, 99, 120, 153, 162, 189, 198, 231, 240, 267, 285, 306,\\
&\quad 324, 333, 357, 369, 375, 378\}, \{22, 34, 37, 49, 82, 88, 136, 145, 148,\\
&\quad 163, 169, 196, 199, 211, 265, 271, 295, 298, 322, 328, 352\}\}\\
C_8 &= \{\{2, 8, 32, 38, 47, 50, 107, 122, 128, 131, 143, 146, 152, 188, 191,\\
&\quad 200, 203, 221, 227, 341, 371\}, \{27, 33, 51, 54, 63, 66, 102, 108, 111,\\
&\quad 123, 126, 132, 147, 204, 207, 216, 222, 246, 252, 264, 294\}\}\\
C_9 &= \{\{14, 23, 56, 65, 92, 101, 134, 155, 173, 194, 224, 239, 257, 260, 266,\\
&\quad 278, 302, 311, 329, 350, 368\}, \{43, 55, 58, 85, 91, 106, 109, 118, 166,\\
&\quad 172, 205, 217, 220, 232, 283, 307, 313, 337, 340, 364, 370\}, \{381\}\}
\end{aligned}
$$

Figure 15: The first cycle of $W(9, 3, 783)$ which was obtained using $PP(381, 7^6)$.

also holds for the largest certificates of unknown van der Waerden numbers. Apart from $W(3, 3)$, there exists largest known certificates with at least two internal symmetries. For $W(3, 3)$ there exist only extreme certificates with a single internal symmetry. We conjecture that apart from the case $W(3, 3)$ there exists extreme certificates with at least two internal symmetries.

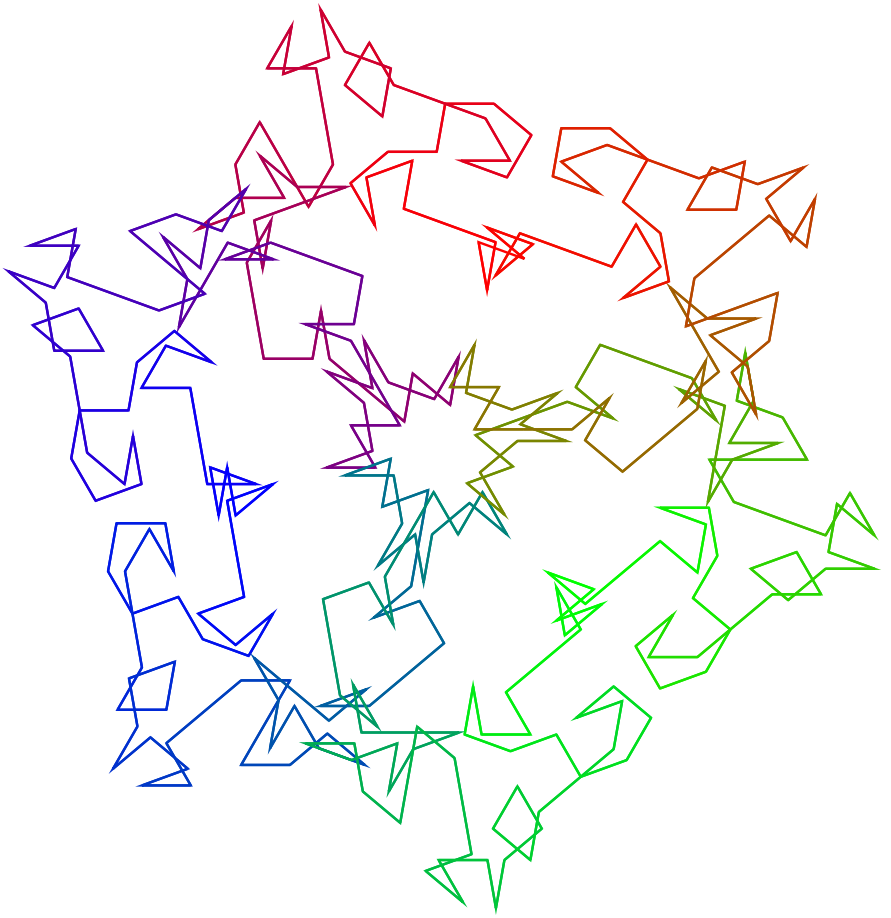

Figure 16: Visualization of "Spiky Rose" showing that $W(9,3) > 763$.

Conjecture 1. *For all $k \geq 2, l \geq 2$ —expect for $k = 3$ and $l = 3$— there exists an extreme certificate of $W(k,l)$ which contains the internal symmetry σ_{+m}.*

It seems plausible that all $W(k,l)$ with $l > 3$ have internal symmetry σ_{+m} as it allows to duplicate a certificate of length m by $l-1$ times.

Conjecture 2. *For all $k \geq 2, l \geq 2$ —expect for $k = 3$ and $l \in \{2,3\}$— there exists an extreme certificate of $W(k,l)$ which contains the internal symmetry σ_{+m} and $\sigma_{\times r^i \circ \mathrm{color}^*}$ for a $i \in \{1,\ldots,k\}$.*

Other internal symmetries are common, but probably not general. For example, during our experiments, we found some extreme and largest known

Table 7: Known van der Waerden numbers and the best known lower bounds. Lower bounds presented in bold are improvements first published in this paper. The number between brackets shows the factor used for pre-partitioning. We denote with $\times$ that the largest known certificate is not based on pre-partitioning

$l\backslash k$	2	3	4	5	6
3	9 $(\times)$	27 $(\times)$	76 (2^4)	$>$ **170** (3^4)	$>$ **223** (2^4)
4	35 (2^2)	293 (5^3)	$>$ 1048 (2^4)	$>$ 2254 (3^5)	$>$ 9778 (3^6)
5	178 (3^2)	$>$ **2173** (10^6)	$>$ 17705 (3^4)	$>$ **98741** (3^4)	$>$ **98748** (3^4)
6	1132 (3^2)	$>$ **11191** (5^6)	$>$ 91331 (7^4)	$>$ **540025** (7^4)	$>$ **816981** (11^4)
7	$>$ 3703 (3^2)	$>$ 43855 (6^3)	$>$ 420216 (11^4)		
8	$>$ 11495 (3^2)	$>$ 238400 (5^3)			
9	$>$ 41265 (11^2)				

$l\backslash k$	7	8	9	10	11	12
3	$>$ **342** (2^6)	$>$ **511** (7^4)	$>$ **763** (7^6)	$>$ **889** (5^6)	$>$ **1183** (2^{14})	$>$ **2329** (5^8)

certificates of $W(k,3)$ with internal symmetry $\sigma_{+p\circ\text{color}}$ — except for $k \in \{3,7,10\}$.

6.2. Lower bounds for triples in arithmetic progression

Several patterns can be observed in certificates. Moreover, there are patterns in the size of the largest known certificates as well. In particular, the size of the largest known certificates of $W(k,3)$ are close to k^3.

Conjecture 3. *For all $k \geq 2$, it holds that $W(k,3) \geq k^3$*

The improved lower bounds presented in this paper support the above lower bound claim for $k \leq 9$. Until now this was only known for $k \leq 5$. Yet for $k > 9$ it remains an open problem.

There is a curious coincidence regarding k^3 lower bounds. For $W(k,3)$ with $k \in \{3,7,8\}$, the best known/possible lower bound is exactly k^3. Furthermore, all certificates of these lower bounds have more in common: the size of the color classes is not equal. On the other hand, for $W(k,3)$ with a lower bound $W(k,3) > k^3$ (i.e., $k \in \{2,4,5,6,9,12\}$), all color classes of the largest known/possible certificates have exactly the same size[3].

Conjecture 4. *For all $k \geq 2$, a certificate $W(k,3,n)$ with $n \geq k^3 - 1$ can be constructed in polynomial time.*

[3]If the elements $i \cdot p_n \pmod n$ with $i \in \mathbb{N}$ are neglected.

Table 8: Internal symmetries in extreme and largest known certificates

k	l	m	n	p	q	r	internal symmetry
2	3	4	8	2	2	1	σ_{+m}, $\sigma_{+p} \circ \text{color}$, $\sigma_{m-} \circ \text{color}$
3	3	13	26	13	1	2	σ_{n-}
4	3	37	75	37	1	2	σ_{+m}, $\sigma_{\times r} \circ \text{color}$, $\sigma_{m-} \circ \text{color}$
5	3	85	170	17	5	3	σ_{+m}, $\sigma_{+p} \circ \text{color}$, $\sigma_{\times r} \circ \text{color}$
6	3	111	223	37	3	2	σ_{+m}, $\sigma_{+p} \circ \text{color}$, $\sigma_{\times r^2} \circ \text{color}$, $\sigma_{m-} \circ \text{color}$
7	3	171	342	19	9	2	σ_{+m}, $\sigma_{\times r^3} \circ \text{color}$, $\sigma_{m-} \circ \text{color}$
8	3	255	511	17	15	7	σ_{+m}, $\sigma_{+p} \circ \text{color}^*$, $\sigma_{\times r} \circ \text{color}^*$
9	3	381	763	127	3	7	σ_{+m}, $\sigma_{+p} \circ \text{color}$, $\sigma_{\times r^2} \circ \text{color}$, $\sigma_{m-} \circ \text{color}$
10	3	444	889	37	12	5	σ_{+m}, $\sigma_{\times r^3} \circ \text{color}$
11	3	591	1183	197	3	2	σ_{+m}, $\sigma_{+p} \circ \text{color}^*$, $\sigma_{\times r} \circ \text{color}^*$
12	3	1164	2329	97	12	5	σ_{+m}, $\sigma_{+p} \circ \text{color}$, $\sigma_{\times r^2} \circ \text{color}$

Even in the case that Conjecture 3 holds, a construction method will be hard to develop. For instance, recall the improved lower bound for $W(7,3)$. Although the found certificate shows several symmetries (such as the pre-partition $PP(171, 2^6)$ and the reflection symmetry in Figure 12), the current generalized construction methods are not able to generate a certificate with the size of the new bound. This is even the case for $W(3,3)$. Therefore, the current challenge regarding this conjecture is to develop an elegant method that generates all existing largest known lower bounds.

6.3. Lower bounds vs upper bounds

Although this paper discussed several patterns occurring in the largest known certificates for van der Waerden numbers, it appears that there is also a relation between the sizes of these largest known certificates. Notice that largest known lower bounds for $W(k,l)$ are close to k^{2l-3}. Table 9 shows a comparison between the numbers. Especially for the lower numbers, k^{2l-3} seems to be an accurate approximation of $W(k,l)$. For the largest numbers, the approximation appears to be larger. That can be explained by the fact that the lower bound techniques that are used for these numbers are not as advanced as those used for the lower numbers. Therefore, it is expected that the lower bounds for the larger numbers can be improved.

As mentioned in the introduction, there exists a vast body of work on upper bounds of van der Waerden numbers. The best known upper bounds of $W(k,l)$ are by Gowers [9]:

$$W(k,l) \leq 2^{2^{k^{2^{2^{l+9}}}}}$$

Table 9: A comparison between the van der Waerden numbers $W(k,l)$ —or their largest known lower bounds— and the approximation function k^{2l-3}

$l\backslash k$	*2*	*3*	*4*	*5*	*6*
2	3	4	5	6	7
	2	3	4	5	6
3	9	27	75	> 170	> 233
	8	27	64	125	216
4	35	293	> 1048	> 2254	> 9778
	32	243	1024	3125	7776
5	178	> 2173	> 17705	> 98740	
	128	2187	16384	78125	
6	1132	> 11191	> 91331	> 540025	
	512	19683	262144	1953125	
7	> 3703	> 48811	> 420217		
	2048	177147	4194304		
8	> 11495	> 238400			
	8192	1594323			
9	> 41265				
	32768				

Our experiments suggest that there is quite some room for improvement. Our final conjecture is that all van der Waerden numbers $W(k,l)$ are bounded from above by k^{2l}.

Conjecture 5. *$W(k,l) < k^{2l}$ for all $k \geq 2$ and $l \geq 2$.*

7. Conclusions

We analyzed extreme and largest known certificates of van der Waerden numbers and observed several symmetries. Especially the rotation symmetry was useful to improve several lower bounds by enforcing this symmetry to the SAT encoding of van der Waerden certificates. Afterwards we analyzed why existing methods were not able to find these lower bounds. Although many of the patterns of existing methods were observable in the new certificates, they were slightly more complex. We presented two new patterns, pre-partitions and internal symmetries, that capture the more complex certificates. Enforcing pre-partitions and/or internal symmetries allowed us to improve more lower bounds of van der Waerden numbers — in particular bounds of $W(k,3)$ with $k \geq 7$.

Apart from patterns in certificates, we observed patterns in the lower bounds of van der Waerden numbers. The lower bounds of van der Waerden numbers $W(k,3)$ are very close to k^3 in the experimented domain of $k \in \{2,\ldots,12\}$: all lower bounds are between $0.85 \cdot k^3$ and $1.7 \cdot k^3$. Only the

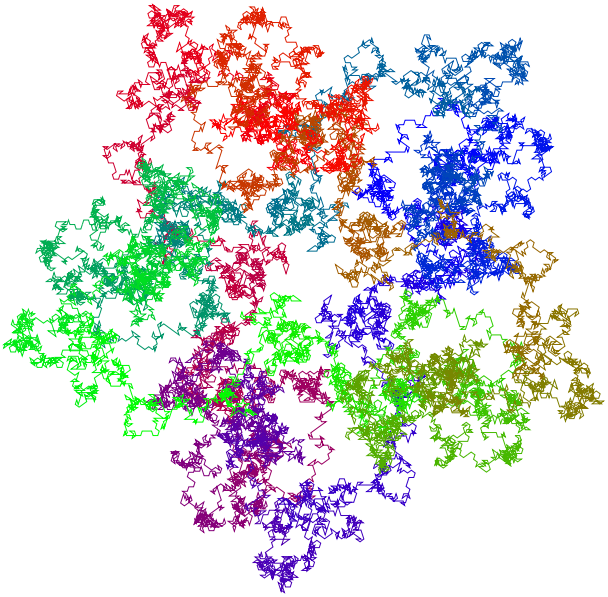

Figure 17: Visualization of a certificate showing that $W(5,5) > 98741$.

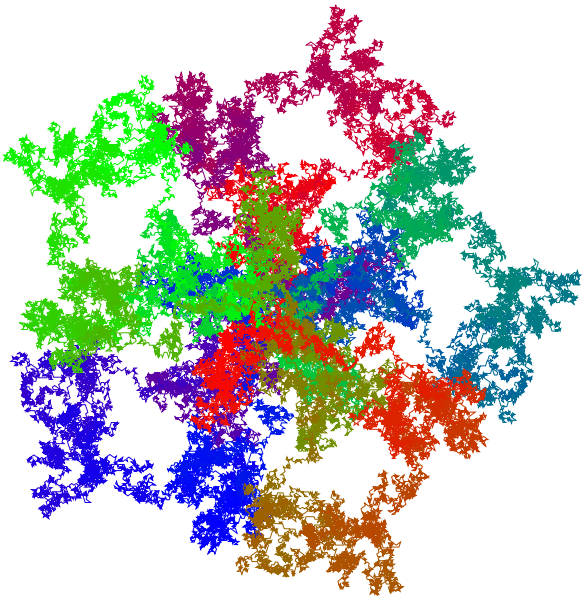

Figure 18: Visualization of a certificate showing that $W(5,6) > 540026$.

lower bounds of $W(10,3)$ and $W(11,3)$ are somewhat below k^3. We expect that this is due to the potential limited success of the current methods and conjecture that $W(k,3) \geq k^3$. Similar patterns can be observed for $W(k,l)$ in general and we conjecture that $W(k,l) < k^{2l}$.

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