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The Solution

A proof long enough to stump Leonhard Euler

Almost identical mathematical problems can result in wildly different solutions. In this piece of counterfactual history, Bernardo Subercaseaux and Marijn Heule discuss two graph theoretic problems that Leonhard Euler did not have the opportunity to meet. While the proof for the first problem can be elegantly summarized into a paragraph, thanks to Euler's results, the shortest proof known for the second problem uses over 30 terabytes of data. Therefore, we argue that despite his mathematical genius, Euler would not have been able to solve the latter problem. The almost alien prolificity of the Swiss mathematician makes him a perfect fictitious intermediary for discussing the divide between the world of human mathematics and the world of computational mathematics.

The life and work of Leonhard Euler (Basel, 1707–St. Petersburg, 1783), arguably one of the greatest mathematicians to ever live, is more than enough to fill up dozens of books. With a grand total of 866 articles, averaging 800 pages per year, and contributions to areas as diverse as differential equations, number theory, and complex analysis, Euler is still the most prolific mathematician to date [6]. His personal life was no less eventful, marked by two marriages, thirteen children, of whom only five reached adulthood, and a series of medical difficulties leading up to complete blindness [1]. The interplay between Euler's poor health and his stoic nature coupled with unmatched work ethic is succinctly described by Eves [9]. In 1735, Euler spent three days and two nights working on a problem on celestial mechanics, and the strain of the effort lead to the permanent loss of vision in his right eye. "Now I will have less distraction", commented Euler, and his productivity indeed increased.

He would go on to lose vision in his left eye due to a cataract in 1766. For a natural comparison, around 1822, Ludwig van Beethoven famously composed his Symphony No. 9 while completely deaf, resulting in one of the most admirable episodes of music history.

Incapable of rendering a fair homage to Euler, this brief article discusses only two mathematical problems, and worse still, it has the audacity to speculate about how Euler would have faced such problems even though he never had the opportunity to see them, as they originated more than 200 years after his death. The two problems, detailed in next section, belong to the area of graph theory, kickstarted in 1735 by Euler himself, and more specifically, to packing colorings over infinite graphs [3]. We argue in one of the following sections that Euler could have comfortably solved the first problem, leveraging his celebrated solution to the Basel problem (1734), and yet, despite his unquestionable math-

ematical genius, we claim that Euler would not have been able to prove a solution to the second problem. We will justify this claim later on in this article. Being far less mathematically talented than the Swiss master, our success in solving the second problem during 2023 could not have been without modern computational tools, taking advantage of both software and hardware beyond the wildest dreams of Euler's 18th century. We hope that our exposition highlights both the power of automated reasoning techniques in mathematics and the blurry line that separates problems that admit elegant and short mathematical arguments from those inherently requiring long computational proofs.



Figure 1 Commemorative stamp issued by the Swiss Post in 2007 to celebrate the 300th anniversary of Euler's birth [14]. The portrait in the stamp was made by Jakob Emanuel Handmann in 1753.



Figure 2 Representation of the Königsberg bridges problem as a graph problem, with an intermediate step in which the topology of the city has been simplified [12, 21].

Graph theory and two coloring problems

The beginning of graph theory is traditionally identified with Euler's solution to the problem of the *Seven Bridges of Königsberg* [6, 19]. In that problem, one is challenged to traverse the seven bridges of the city of Königsberg (see the left of Figure 2), passing exactly once per bridge. Euler proved in 1735 that no solution existed by analyzing the parity of the number of bridges touching a landmass, and generalized his observation to arbitrary cities with any number of landmasses and bridges. The main insight of his generalized solution was simplifying the complex geometry of the city into an abstract model of the relationship between physical landmasses and bridges.

Each landmass would correspond to a dimensionless object called *vertex*, and bridges connecting landmasses would, in turn, correspond to pairs of vertices called *edges*, usually depicted by circles and lines respectively, as on the right of

Figure 2. A *walk* through a graph that passes through every edge exactly once is now called an *Eulerian circuit* in his honor.

Another fundamental family of problems in graph theory, more central to this article, is that of coloring problems, which we introduce next. Due to his work in cartography, Euler would have immediately grasped the idea of *properly coloring a map*; that is, assigning colors to countries (or other geographical demarcations) on a map in such a way that neighboring countries are assigned different colors. Figure 3 illustrates how such a cartographical problem is also a graph problem, where colors are mathematically represented by positive integers.

Formally, we can state that two vertices u, v receiving the same color cannot be connected by an edge, and thus their distance (defined as the length of the shortest edge path connecting them) must be greater than 1.

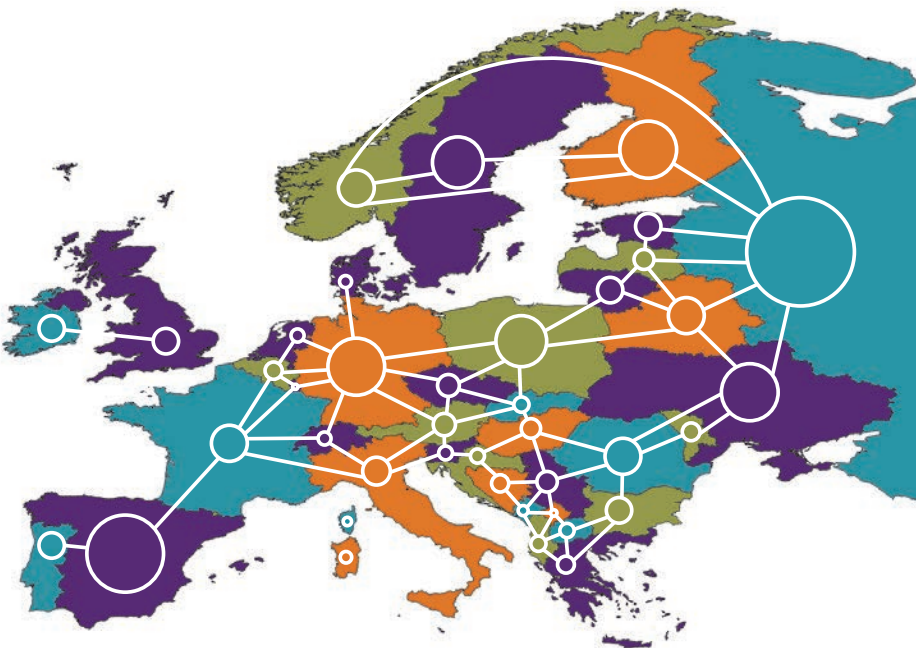


Figure 3 Graph representation of the map coloring problem depicting most of Europe. The fact that four colors are enough for this map is an application of the celebrated *Four Color Theorem* of Appel and Haken, a cornerstone of computer-aided mathematics [20].

Definition. Given a graph $G = (V, E)$, a function $f: V \rightarrow \{1, \dots, k\}$ is a (standard) coloring of G if every pair of vertices $u, v \in V$ that get assigned the same color $c = f(u) = f(v)$ holds $d(u, v) > 1$.

A minor tweak to this definition results in a fascinating family of combinatorial problems. If one changes the distance restriction $d(u, v) > 1$ (of vertices sharing color c) to $d(u, v) > c$, then the distance restriction becomes dependent on the shared color, with larger colors being more restrictive. This alternative notion of coloring was originally motivated by the problem of assigning broadcast frequencies to radio stations; two different stations sharing the same frequency would cause interference unless those stations were ‘far enough’ from each other, where the distance requirement could itself be a function of the assigned frequencies. Under that motivation, Goddard et al. introduced *broadcast colorings* in 2002 [13], which were later rebranded as *packing colorings* [3].

Definition. Given a graph $G = (V, E)$, a function $f: V \rightarrow \{1, \dots, k\}$ is a packing coloring of G if every pair of vertices $u, v \in V$ that get assigned the same color $c = f(u) = f(v)$ holds $d(u, v) > c$.

Naturally, both in standard and packing colorings, one wishes to use the least number of colors for a given graph, which is said to be its (packing) chromatic number. For a given graph G , we use notation $\chi(G)$ for its chromatic number and $\chi_\rho(G)$ for its packing chromatic number. An example illustrating the difference between standard graph colorings and packing colorings is presented in Figure 4, showing that if G is the 3×3 grid graph, then $\chi(G) = 2$ and $\chi_\rho(G) = 4$.

We are now ready to present the two main problems considered in this article, which concern the packing chromatic number of two infinite graphs. First, the graph \mathbb{Z}^2 , often called the infinite square grid, or the infinite square lattice, has all pairs of integers as vertices and edges between pairs $(a, b), (c, d)$ when $|a - c| + |b - d| = 1$. Then, \mathbb{Z}_{Ch}^2 is the infinite Chebyshev grid, whose vertex set is also \mathbb{Z}^2 , but has also edges between diagonally adjacent pairs of vertices, i.e., when $\max(|a - c|, |b - d|) = 1$. The two graphs are illustrated in Figure 5.

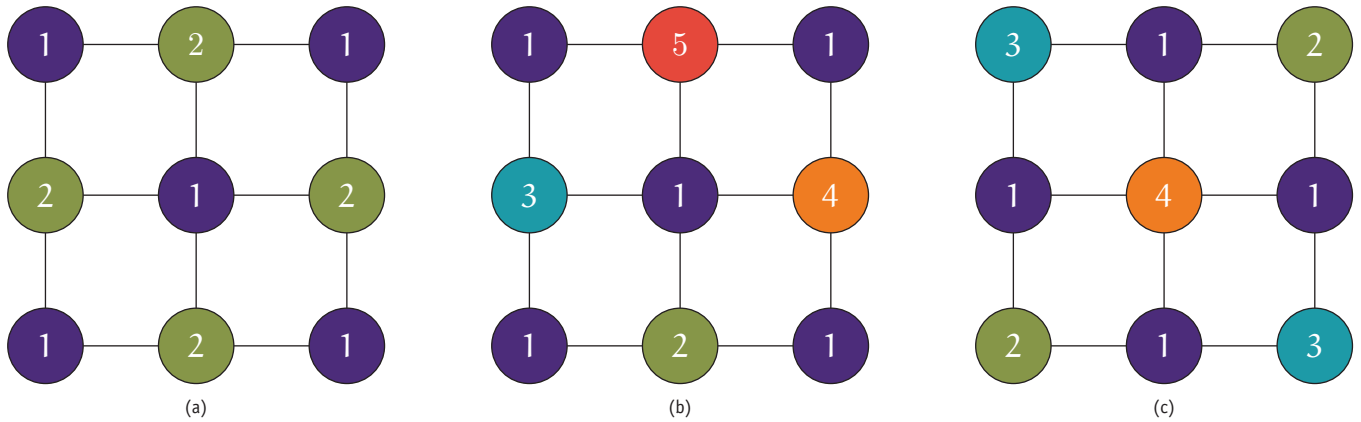


Figure 4 Illustration of standard and packing colorings for the 5×5 grid graph. (a) A standard coloring of the 3×3 grid graph. (b) A packing coloring of the 3×3 grid graph using five colors. (c) An optimal packing coloring of the 3×3 grid graph using four colors.

As suggested by Figure 5, the addition of diagonal edges makes the packing chromatic number of \mathbb{Z}_{Ch}^2 larger than that of \mathbb{Z}^2 , but the exact difference is far from obvious: the packing chromatic number of \mathbb{Z}^2 is 15, and \mathbb{Z}_{Ch}^2 does not admit any packing coloring using a finite number of colors. Finbow and Rall proved in 2010 that $\chi_\rho(\mathbb{Z}_{\text{Ch}}^2) = \infty$ [11] through a cute *density-based* argument we mimic in the next section. (The result of Finbow and Rall [11, Theorem 6] is slightly stronger, as it concerns the infinite triangular lattice.) On the other hand, the packing chromatic number of \mathbb{Z}^2 resisted twenty years of incremental work; and was deemed the most important among the pack-

ing coloring problems for infinite graphs [3], until being solved by the authors in 2023 [18].

Density arguments

Let us start with a finite example that we will then extend to infinite graphs. Consider a finite graph $G = (V, E)$ and a packing coloring f for G that uses colors $\{1, 2, 3, 4\}$. Then, for each color $c \in \{1, 2, 3, 4\}$ we can define its *density* (under f) as the proportion of the vertices of G that get color c . (The definition of *density* used in this section was introduced by Fiala et al. [10].) For example, in the bottom of Figure 4, the density of color 1 is $\frac{4}{9}$, while for color 4 it is only $\frac{1}{9}$. More formally:

Definition. For a finite graph G , a packing coloring f and a color c , we define

$$d_G(f, c) := \frac{|\{v \in V(G) \mid f(v) = c\}|}{|V(G)|}.$$

Naturally, we can define the maximum possible density of a given color c as $d_G(c) = \max_f d_G(f, c)$.

It is direct from the definition that $\sum_{c \in \text{range}(f)} d_G(f, c) = 1$ for any packing coloring f , and as a result we get the following simple lemma:

Lemma (folklore). For any graph G , if the densities of a set of colors add up to less than 1, as e.g., $(\sum_{c=1}^k d_G(c)) < 1$, then such a set of colors is not enough for a packing coloring of G , i.e., $\chi_\rho(G) > k$.

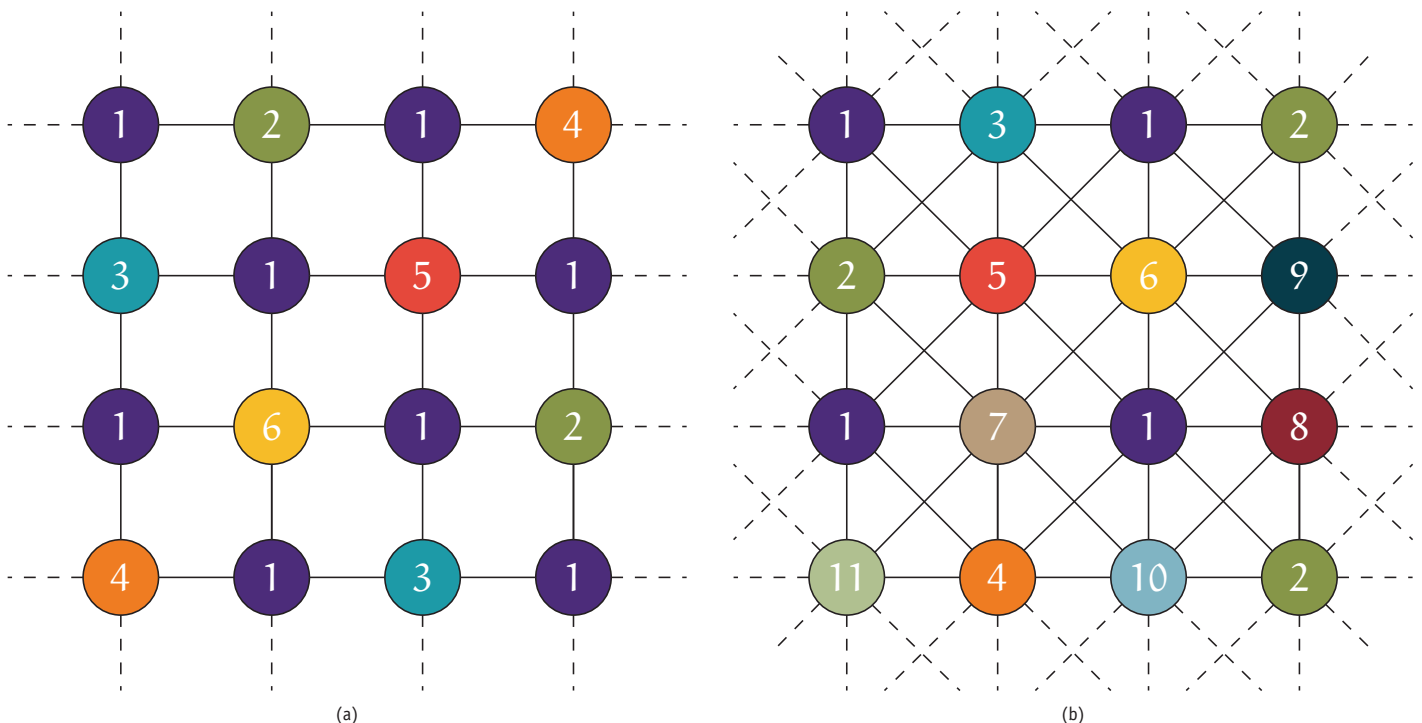


Figure 5 Illustration of \mathbb{Z}^2 , the infinite square grid (a) and \mathbb{Z}_{Ch}^2 , the infinite Chebyshev grid (b).

If we define $d_G(\{1,2\})$ as the maximum fraction of the graph that can get color 1 or 2 over all the packing colorings of G , we obtain the more nuanced notion of *joint density*, where $d_G(\{1,2\}) \leq d_G(1) + d_G(2)$, and this generalizes to any set of colors. The following example shows how this lemma, and its strengthening with joint densities, can be used for proving lower bounds on $\chi_\rho(\cdot)$.

Example. We can use a *density* argument to show that if G is the 3×3 grid graph, then $\chi_\rho(G) = 4$. Indeed, Figure 4(c) shows that $\chi_\rho(G) \geq 4$. Observe that color 1 can be used at most 5 times in G (as in Figure 4(b)), thus showing $d_G(1) = \frac{5}{9}$. Similarly, we have $d_G(2) = \frac{2}{9}$ (as in Figure 4(c)). By the lemma we can already deduce that $\chi_\rho(G) > 2$, as 1 and 2 achieve a density of at most $\frac{7}{9}$. Interestingly, $d_G(3) = \frac{2}{9}$, which suggests that colors $\{1,2,3\}$ would be enough for G . This is not the case, and we will need to consider the *joint density* of 1 and 3, $d_G(\{1,3\})$. We claim that $d_G(\{1,3\}) \leq \frac{6}{9}$. Assume for a contradiction that $d_G(\{1,3\}) \geq \frac{7}{9}$, and then observe that by the pigeonhole principle there must be a *row* of G in which the three vertices get a color in $\{1,3\}$. The only possibility is $(1,3,1)$, in left-to-right order, after which no other vertex can get color 3. As only a single vertex will get color 3, and $d_G(f,1) \leq \frac{5}{9}$ for any f , we conclude $d_G(\{1,3\}) \leq \frac{6}{9}$. Putting this together with $d_G(2) \leq \frac{2}{9}$ yields that $\chi_\rho(G) > 3$ by the natural generalization of the lemma to include joint densities.

The idea of density can be extended to infinite graphs by considering the limiting density of increasingly larger finite subgraphs (e.g., balls of increasing radius around a vertex), and the lemma holds as well for infinite graphs. Let us see how this is useful for studying $\chi_\rho(\mathbb{Z}_{\text{Ch}}^2)$. Consider a $(k+1) \times (k+1)$ subgraph of \mathbb{Z}_{Ch}^2 (e.g., bottom of Figure 5 depicts a 4×4 subgraph) and note that because of the diagonal edges at most one vertex in the subgraph can get color k (e.g., in the bottom of Figure 5 at most one vertex can get color 3). In general, this implies

$$d_{\mathbb{Z}_{\text{Ch}}^2}(c) \leq \frac{1}{(c+1)(c+1)} = \frac{1}{(c+1)^2}.$$

To derive a lower bound from this and the lemma, we will have to find the largest

possible k such that $\sum_{c=1}^k d_{\mathbb{Z}_{\text{Ch}}^2}(c) < 1$, and equivalently, $\sum_{c=1}^k \frac{1}{(c+1)^2} < 1$. The next section is devoted to answering this question.

The Basel problem

The infinite sum of the reciprocals of the integers, namely $1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$, was already known to diverge in the fourteenth century, by proof of Nicole Oresme. On the other hand, the sum of the reciprocals of the squares,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

was explicitly posed as a problem for the first time by Pietro Mengoli in 1650. This infinite sum stumped prominent mathematicians until it finally fell to the hand of Euler in 1734, bringing him immediate fame as a 28 year old prodigy. The value of this infinite sum, now known as the *Basel problem*, perhaps surprisingly, turns out to involve the circle constant π ,

Theorem. *The infinite sum of the reciprocals of the squares of the natural numbers, $\sum_{n=1}^{\infty} 1/n^2$, equals $\pi^2/6$.*

The proof of this theorem is a beautiful example of Euler’s style, which we will illustrate next. Geniality, has been said, appears when reconciling two seemingly unrelated, or even contradictory ideas [17]; the genius of Isaac Newton (1643–1727), for example, was to notice that the apple that falls, and the moon that does not fall, in superficially opposite behaviors, are in fact both instances of the same underlying phenomenon [1, 17]. In a similar fashion, Euler’s solution to the Basel problem works by reconciling two different perspectives on the $\sin(x)$ function.

Proof. On the one hand, the Maclaurin series, well known at the time, allows expressing $\sin(x)$ as an infinite polynomial:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

and consequently,

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots,$$

where now x^2 appears, and with a coefficient of $-\frac{1}{3!} = -\frac{1}{6}$. On the other hand, consider for example that we have a polynomial P of degree 3 with roots $\{-2, 1, 3\}$, and

we want to obtain the coefficient of x^2 in P provided that the constant term in P (i.e., the coefficient of x^0) is 1. Then we simply write P in terms of its roots and a constant factor α , as

$$P(x) = \alpha \cdot (x+2)(x-1)(x-3) = \alpha \cdot (x^3 - 2x^2 + 5x + 6),$$

which implies $\alpha = \frac{1}{6}$ as the constant term of $P(x)$ must be 1, and thus the coefficient of x^2 is $-\frac{2}{6}$. Euler used this same form of reasoning to $\sin(x)$, treating it as an infinite polynomial. Euler knew that $\sin(k \cdot \pi) = 0$ for every integer k , and assumed (correctly but without proof), that those were the only roots of the \sin function. Naturally, that would imply that the only roots of $\frac{\sin(x)}{x}$ are $\{k \cdot \pi \mid k \in \mathbb{Z}, k \neq 0\}$. Thus, Euler felt justified in writing

$$\frac{\sin(x)}{x} = \alpha \cdot (x + \pi)(x - \pi)(x + 2\pi)(x - 2\pi) \cdot (x + 3\pi)(x - 3\pi) \dots$$

Now, notice that $(x + n\pi) = n\pi(1 + \frac{x}{n\pi})$, and thus $\frac{\sin(x)}{x}$ can be rewritten altering its constant factor as:

$$\begin{aligned} \frac{\sin(x)}{x} &= \alpha' \cdot \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{2\pi}\right) \dots \\ &= \alpha' \cdot \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots \end{aligned}$$

Because of the well-known limit $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$, one immediately gets that $\alpha' = 1$. Thus, Euler arrived at an infinite expression for $\sin(x)$ alternative to its Maclaurin series. In this alternative expression, if we were to multiply all the parentheses to get a polynomial, the only terms with x^2 would be of the form $\frac{-x^2}{k^2\pi^2} \cdot 1 \cdot 1 \dots$ (i.e., quadratic terms multiplied by all the other constant terms). Therefore, the coefficient of x^2 in the whole expression for $\sin(x)/x$ is given by

$$\begin{aligned} &-\frac{1}{\pi^2} + \frac{-1}{4\pi^2} + \frac{-1}{9\pi^2} + \frac{-1}{16\pi^2} + \dots + \frac{-1}{n^2\pi^2} \\ &= \left(\frac{-1}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

As this coefficient must be equal to the coefficient of x^2 in the Maclaurin series, $-\frac{1}{6}$, one gets

$$\left(\frac{-1}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{6},$$

from where Euler correctly concluded the answer to the Basel problem must be exactly $\pi^2/6$. \square

The proof that Euler would have nailed

We believe that Euler could have easily determined $\chi_\rho(\mathbb{Z}_{\text{Ch}}^2)$ by combining the observations about the density arguments with his celebrated solution of the Basel problem presented above. Indeed, after arriving at $d_{\mathbb{Z}_{\text{Ch}}^2}(c) \leq \frac{1}{(c+1)^2}$ as we did before, we believe the Swiss master would have immediately noticed that

$$\begin{aligned} \sum_{c=1}^{\infty} d_{\mathbb{Z}_{\text{Ch}}^2}(c) &\leq \sum_{c=1}^{\infty} \frac{1}{(c+1)^2} \\ &= \left(\sum_{c=1}^{\infty} \frac{1}{c^2} \right) - 1 \\ &= \frac{\pi^2}{6} - 1 \approx 0.644 < 1, \end{aligned}$$

implying that no finite amount of colors is enough for a packing chromatic of \mathbb{Z}_{Ch}^2 , and thus $\chi_\rho(\mathbb{Z}_{\text{Ch}}^2) = \infty$.

Finbow and Rall [11] proved a theorem that implies $\chi_\rho(\mathbb{Z}_{\text{Ch}}^2) = \infty$ using a telescopic argument resembling:

$$\begin{aligned} \sum_{c=1}^{\infty} d_{\mathbb{Z}_{\text{Ch}}^2}(c) &\leq \sum_{c=1}^{\infty} \frac{1}{(c+1)^2} \\ &< \sum_{c=1}^{\infty} \frac{1}{c \cdot (c+1)} \\ &= \sum_{c=1}^{\infty} \left(\frac{1}{c} - \frac{1}{c+1} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{c=1}^n \left(\frac{1}{c} - \frac{1}{c+1} \right) \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1. \end{aligned}$$

Albeit much simpler, this argument provides a weaker bound than the one derived by following Euler’s work, as from the Basel problem we deduce that is not even possible to packing color 65% of \mathbb{Z}_{Ch}^2 . As it will be shown in the next section, packing coloring density arguments can require significant precision, which makes the understanding of infinite density-based sums a key technique in the area.

The computational nature of $\chi_\rho(\mathbb{Z}^2)$

Interestingly, there does not seem to exist a simple argument for determining $\chi_\rho(\mathbb{Z}^2)$. If we attempt to repeat the same argument presented in the previous section, we get the significantly weaker bound

$$d_{\mathbb{Z}^2}(c) \leq \frac{2}{c^2}, \tag{1}$$

as every c can be used at most in two opposite corners of a $c \times c$ subgrid (cf. the top of Figure 5 where color 4 occurs twice and could not be used any further). This bound is vacuous for $c = 1$, where it is easy to see that $d_{\mathbb{Z}^2}(1) = \frac{1}{2}$. The bound implies only that $d_{\mathbb{Z}^2}(2) \leq \frac{1}{2}$, which could even suggest that colors 1 and 2 are enough for \mathbb{Z}^2 ! However, analyzing a 3×3 subgrid of \mathbb{Z}^2 shows that $d_{\mathbb{Z}^2}(\{1, 2\}) \leq \frac{6}{9}$ (cf. Figures 4(b) and 4(c)). As we get $d_{\mathbb{Z}^2}(3) \leq \frac{2}{9}$ by Equation (1), we can deduce that $d(\{1, 2, 3\}) \leq \frac{8}{9}$, and thus $\chi_\rho(\mathbb{Z}^2) \geq 4$. However $\frac{8}{9} + \frac{2}{4} > 1$,

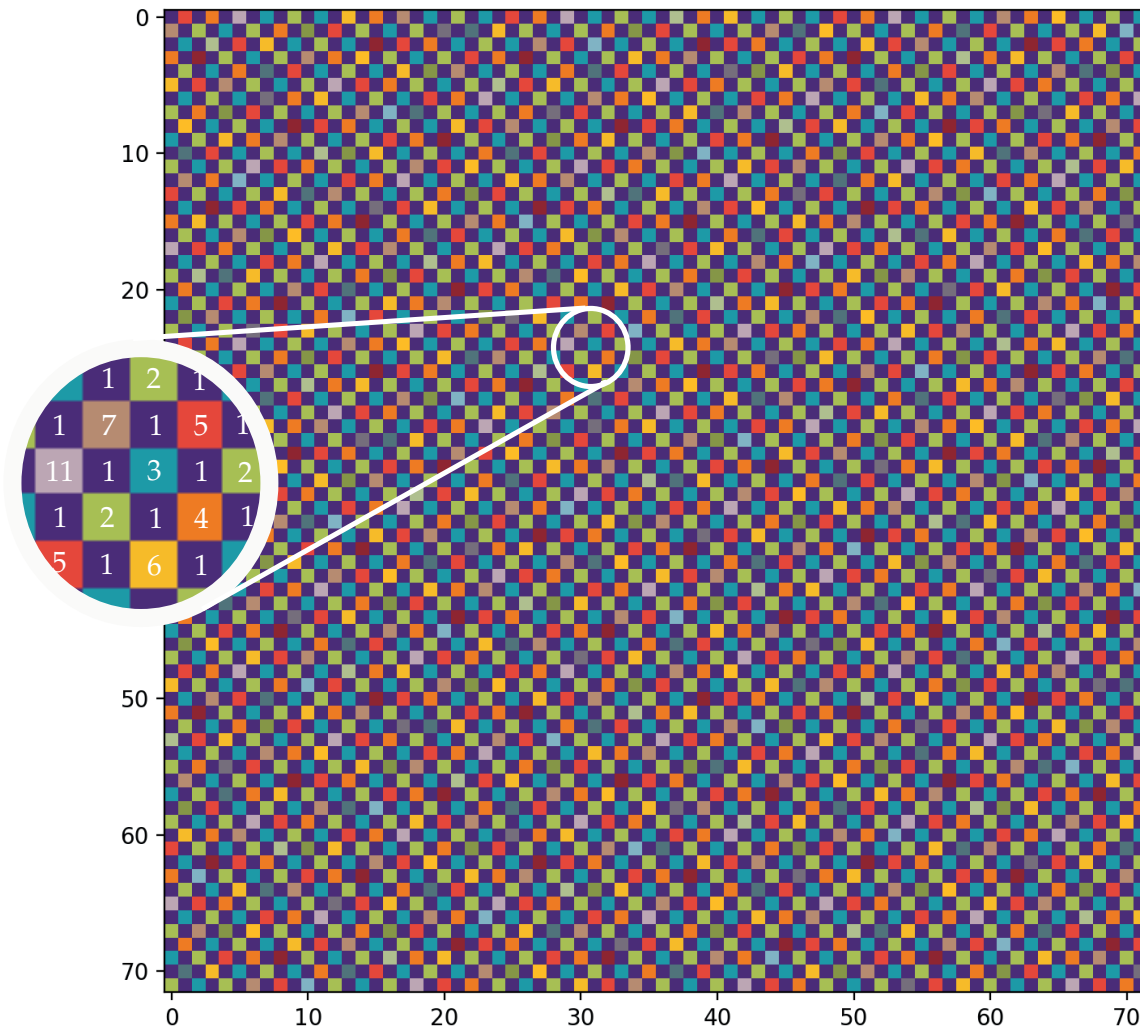


Figure 6 A packing coloring of a 72×72 subgraph of \mathbb{Z}^2 using 15 colors. This same packing coloring can be extended to the infinite graph \mathbb{Z}^2 by stitching it to copies of itself vertically and horizontally. This illustration represents each vertex as a square cell, and the edges between orthogonally adjacent cells are left implicit.

and once again it seems like a small number of colors could be enough for a packing coloring of \mathbb{Z}^2 . By performing more and more careful analyses of the joint densities of small colors, one can get slightly better lower bounds. For instance, in 2009 Fiala et al. [10] used a non-trivial and detailed argument to show that

$$d_{\mathbb{Z}^2}(\{1, 2, \dots, 9\}) \leq \frac{3830381}{3837600} < 1,$$

thus implying that $\chi_\rho(\mathbb{Z}^2) \geq 10$. We claim that Euler displayed enough mathematical talent throughout his life to allow us to infer not only that he would have also reached the bound of Fiala et al., but also

that he could have reached $\chi_\rho(\mathbb{Z}^2) \geq 11$ or even $\chi_\rho(\mathbb{Z}^2) \geq 12$. In terms of upper bounds, we believe it is within the realm of possibility that Euler could have arrived at the right answer, namely that $\chi_\rho(\mathbb{Z}^2) \leq 15$. We already know the example of Isaac Grosz, now a postdoc at Georgia Tech, who was able to deduce an upper bound of $\chi_\rho(\mathbb{Z}^2) \leq 23$ by pen and paper, and as “[Euler] was a fabulous mental calculator, able to perform intricate arithmetical computations without the benefit of pencil and paper.” [6], we believe Euler could have done the same and even further. In fact, the proof of $\chi_\rho(\mathbb{Z}^2) \leq 15$ fits in this brief arti-

cle, displayed in Figure 6. Nonetheless, we believe not even Euler could have proved that $\chi_\rho(\mathbb{Z}^2) \geq 15$ without a computer, and perhaps not even that $\chi_\rho(\mathbb{Z}^2) \geq 12$, as we argue next. The first proof of $\chi_\rho(\mathbb{Z}^2) \geq 12$, by Ekstein et al. [7], used a computer program to explore all possible ways of coloring a finite subgraph of \mathbb{Z}^2 with 11 colors, and after 120 days of computation, having analyzed 43112312093324 configurations (as some symmetries were broken), returned that none of them could work. With today’s methods, a modern computer can prove the lower bound $\chi_\rho(\mathbb{Z}^2) \geq 12$ in less than a second [18], and yet even

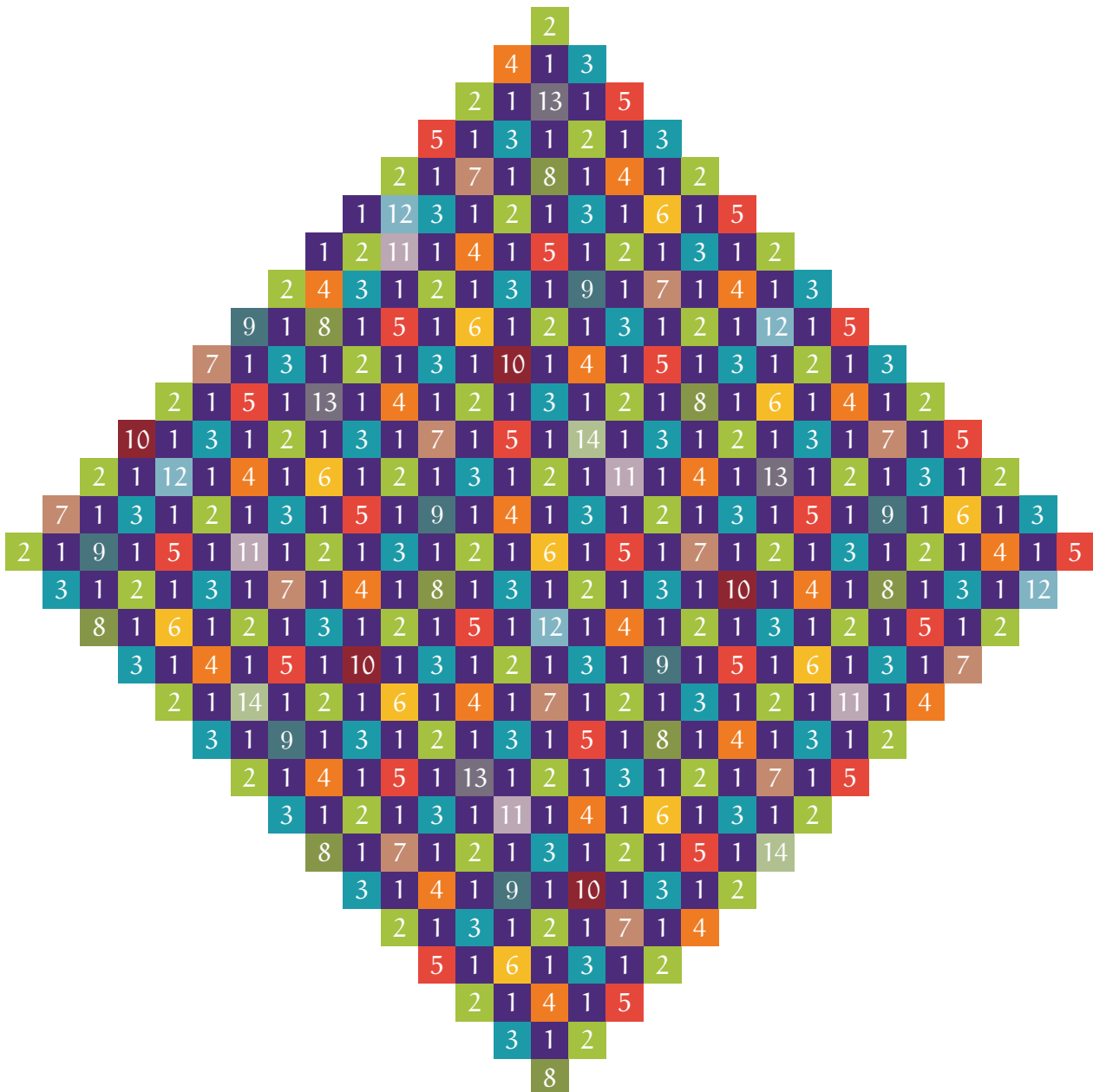


Figure 7 The smallest graph that admits a packing coloring with 14 color, but where no 14-packing coloring exists with a chessboard pattern of 1’s. Note in particular that the north-west outer border of the depicted graph deviates from the chessboard pattern in two vertices.

with such techniques it required over 4000 hours of CPU time to prove $\chi_\rho(\mathbb{Z}^2) \geq 15$, with the resulting proof occupying over 30 terabytes of data, after compression [18]. (For a comparison, the proof of the upper bound of 15, depicted in Figure 6, can be compressed into less than 1 kilobyte.) Moreover, it follows from [18, Section 5] that the smallest finite subgraph of \mathbb{Z}^2 that needs to be considered to observe that $\chi_\rho(\mathbb{Z}^2) > 14$ has at least 365 vertices. In other words, over every subgraph with at most 365 vertices, the joint densities of colors $\{1, \dots, 14\}$ achieve a sum of 1. It is only when considering a subgraph of 421 vertices that it is revealed that 14 colors are not enough for \mathbb{Z}^2 . In the beautiful eulogy that Marquis de Condorcet wrote for Euler in 1783 [5], the year of the master's death, he recounts how Euler once arbitrated a dispute between two of his students, whose calculations up to the 17th term of a complicated series disagreed by a single number. Euler gave his answer after doing the entire calculation in his head; and was later proved to be correct. Similar stories depict Euler as a human computer, able to perform intricate calculations almost instantaneously. We believe, nonetheless, that certain problems do not admit concise proofs (e.g., analyzing fewer than a million cases), and thus computation is a necessary tool for their resolution. Interestingly, as the previous discussions have shown, it is hard to anticipate in advance which

problems will require extensive computation, and which problems will fall to a short and beautiful proof, as the one Euler gifted us.

Conjectures and computer verification

Interestingly, Euler was not fully satisfied with his first proof of 1735, presented above, as it relied on assumptions he could not justify at the time [6]. Euler assumed what is now known as *Weierstrass Factorization Theorem*, namely that factorization and manipulation of finite polynomials in terms of their roots, which were known since Newton, still held for infinite polynomials [6]. In 1741, Euler was able to give a completely different proof of the same result, dissipating the lingering doubts. It took around 100 years for a rigorous proof of the product formula for $\sin(x)/x$ that validated the original 1735 proof. Conjectures sometimes turn out to be false even if they appear intuitively true at first. This is even the case for mathematicians of the caliber of Euler, who incorrectly conjectured the non-existence of Greco-Latin squares of order $(4n+2)$ for $n \geq 1$ [14], or the sum of powers conjecture, disproved in 1966 by a computer search [15]. We believe this is not a weakness of Euler, but rather a testament to a mode of collaboration between mathematicians and computers in which computers help verify or refute conjectures stemming from human intuition that require analyzing large numbers

of cases. As a concluding example, during our work in the packing chromatic number of \mathbb{Z}^2 , we conjectured that color 1 could be assumed without loss of generality to form a *chessboard* pattern, as displayed in Figures 4 to 6. This *chessboard* conjecture seemed very natural as it allowed using color 1 with its maximum possible density, $\frac{1}{2}$, and matched all our experiments over small graphs. Remarkably, the chessboard conjecture turns out to be false, and we have found the smallest counterexample (see Figure 7). It is a graph of 365 vertices that does not admit a 14-color packing coloring if the chessboard pattern of 1s is assumed, and yet it does have a 14-packing coloring when slightly deviating from the pattern.

“The properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstrations. There are many properties of the numbers with which we are well acquainted, but which we are not yet able to prove; only observations have led us to their knowledge. Hence we see that in the theory of numbers, which is still very imperfect, we can place our highest hopes in observations; they will lead us continually to new properties which we shall endeavor to prove afterward.”

Leonhard Euler, 1761 [8, 16]



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