

PRuning Through Satisfaction

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Martina Seidl, and Armin Biere

UT Austin, Vienna University of Technology, and JKU Linz



Haifa Verification Conference

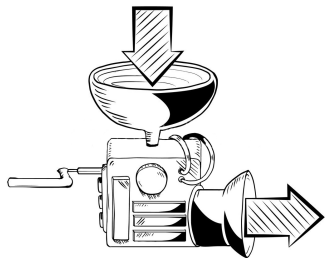
November 15, 2017

Satisfiability Solving (Highly Simplified)

SAT problem:

Given a propositional formula, is it satisfiable?

$$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$$



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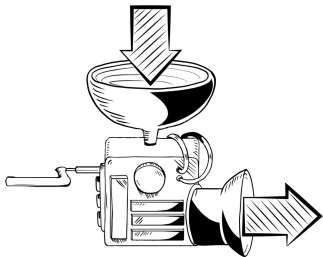
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Input Formula in CNF



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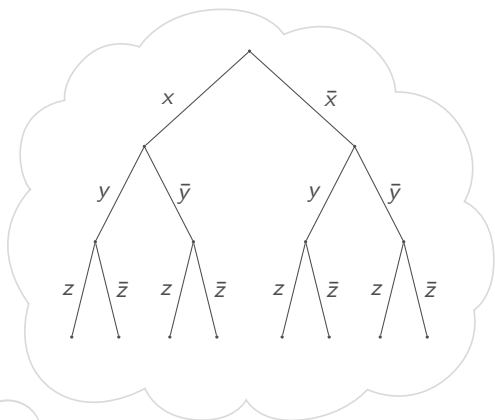
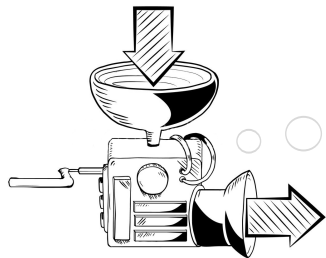


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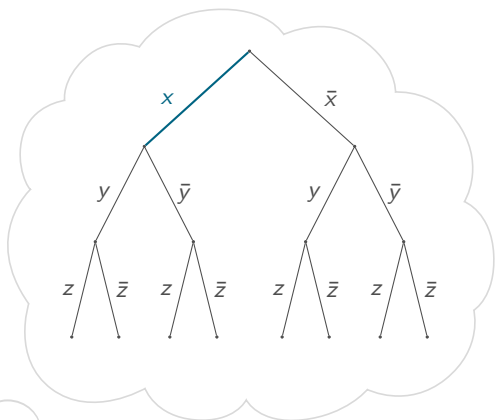
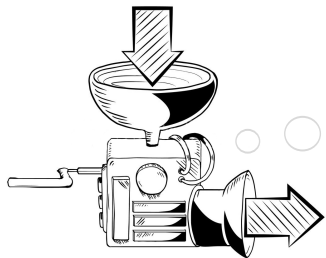


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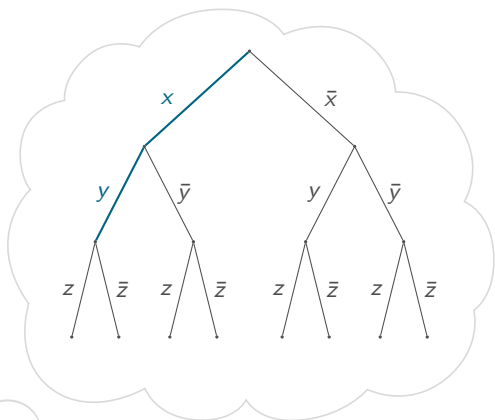
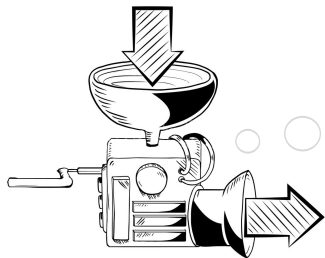


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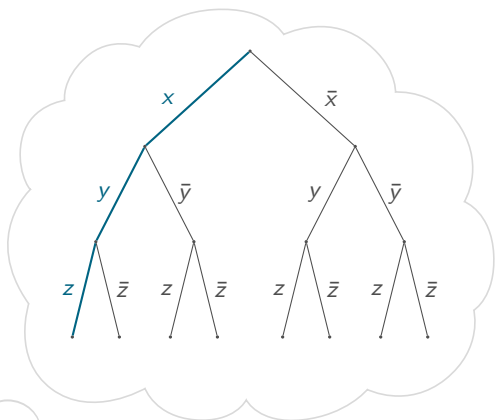
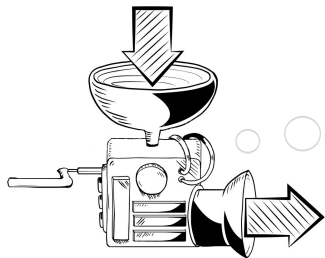


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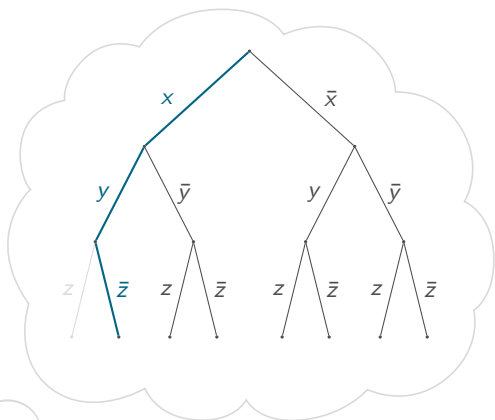
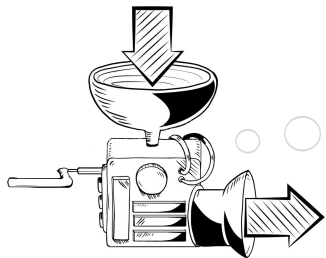


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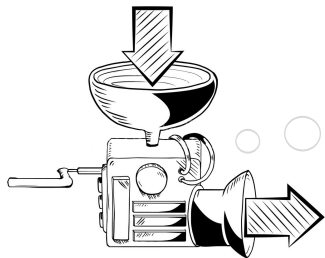


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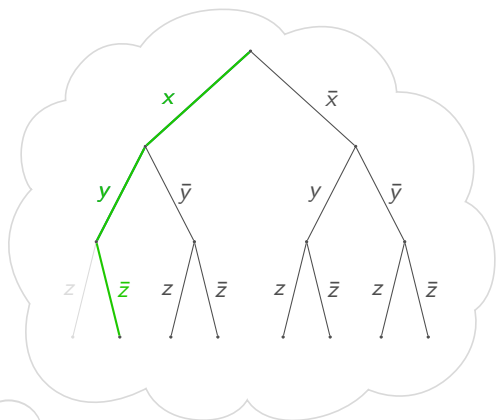
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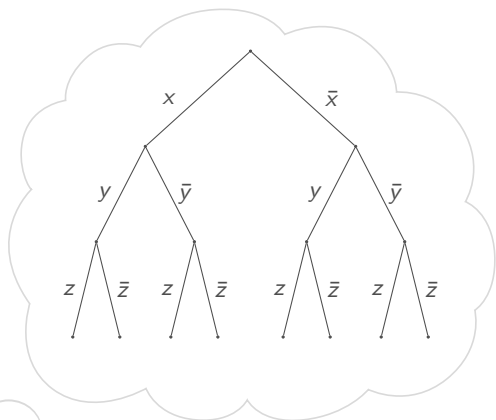
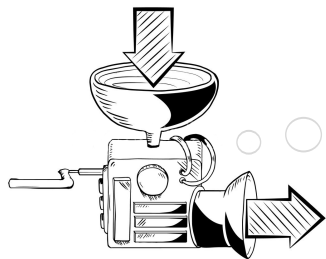
Satisfiable



Key Idea: Prune Less Satisfiable Branches

Can we **prune** earlier?
Even satisfiable branches?

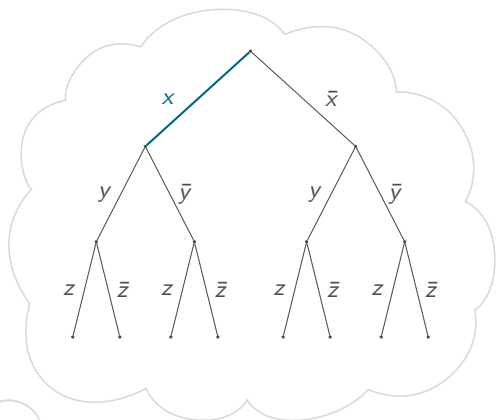
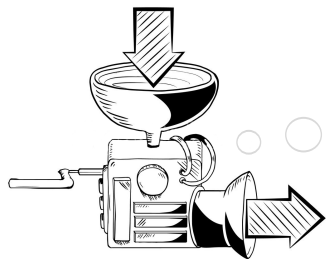
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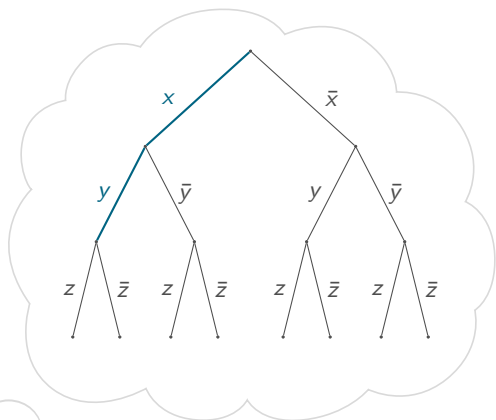
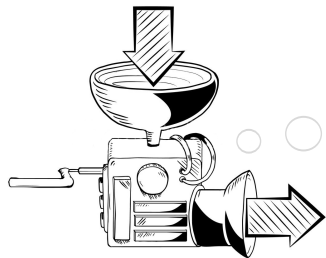
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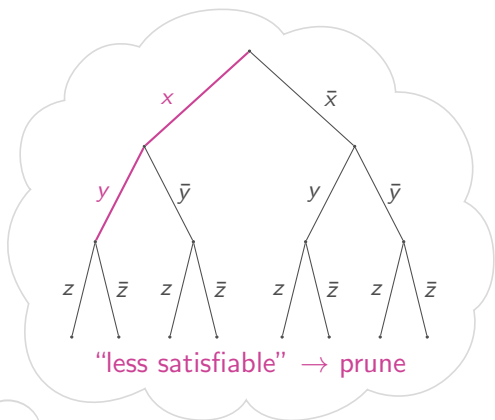
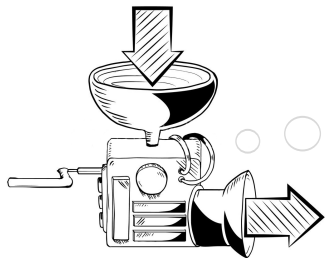
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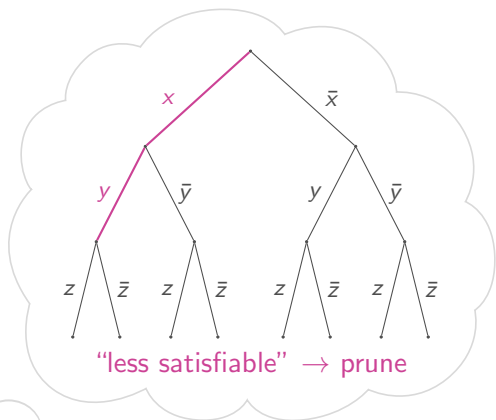
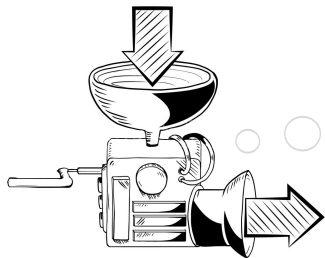
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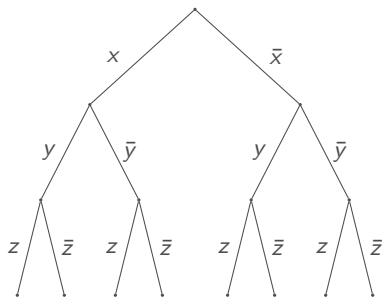
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How to prune? Add **redundant** clauses!

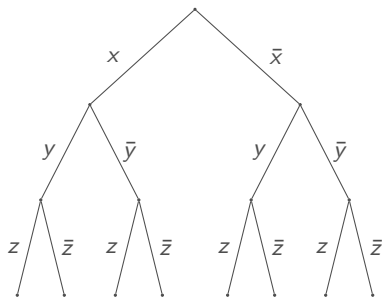
Pruning via Clause Addition

- A clause prunes all branches that falsify the clause.



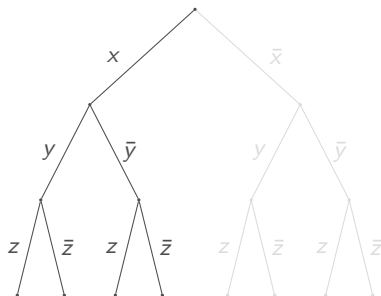
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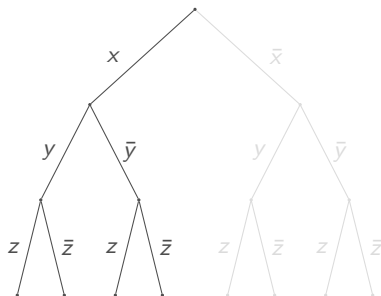
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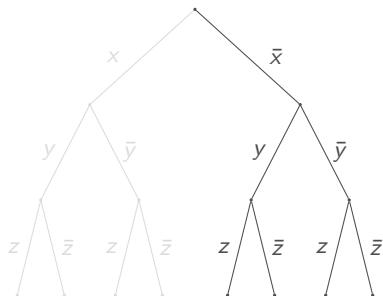
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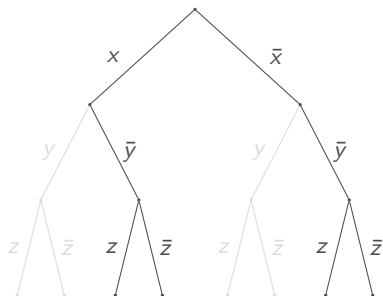
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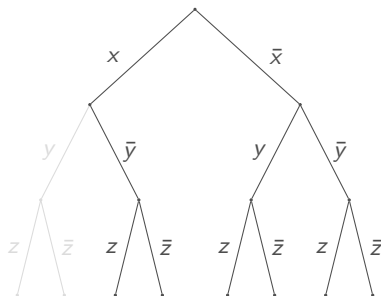
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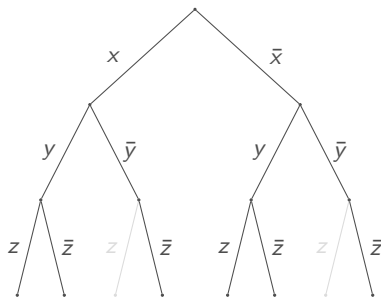
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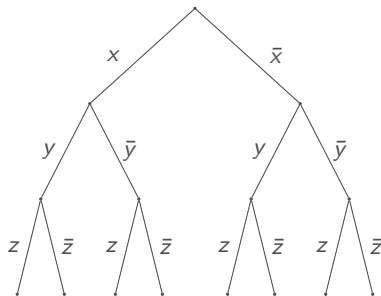
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Introduction

The Positive Reduct

Conditional Autarkies

The Algorithm

Evaluation

Conclusions and Future Work

The Positive Reduct

Traditional Proofs vs. Interference-Based Proofs

- In **traditional** proof systems, everything that is **inferred**, is **logically implied** by the premises.

$$\frac{C \vee x \quad \bar{x} \vee D}{C \vee D} \text{ (res)} \qquad \frac{A \quad A \rightarrow B}{B} \text{ (mp)}$$

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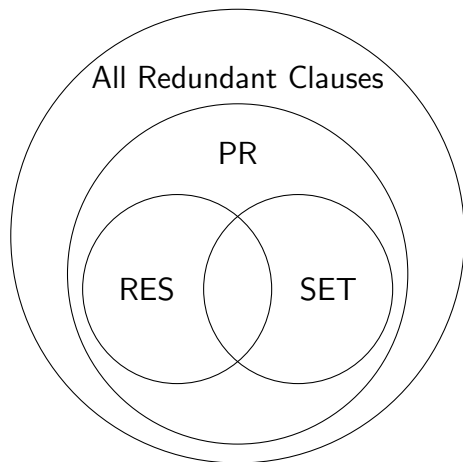
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- ➔ Inference rules reason about the **presence** of facts.
 - If certain premises are present, infer the conclusion.
- **Different approach**: Allow **not only implied conclusions**.
 - **Require only** that the addition of facts preserves **satisfiability**.
 - Reason also about the **absence** of facts.
- ➔ This leads to **interference-based proof systems**.

Redundant Clauses

A clause C is **redundant** w.r.t. a formula F if and only if F and $F \wedge C$ are either both satisfiable or both unsatisfiable.



PR = Propagation Redundant Clauses [CADE'17]

RES = Resolvents

SET = Set-Blocked Clauses [IJCAR'16]

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is SET or PR w.r.t. a formula F is an NP-complete problem.

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Key Idea: While solving a formula F , check whether the positive reduct of F and the current assignment α is **satisfiable**. In that case, **prune** the branch α .

The Positive Reduct: An Example

Given a formula F and a clause C . Let α denote the smallest assignment that falsifies C . The **positive reduct** of F and α , denoted by $p(F, \alpha)$, is the formula that contains C and all *assigned*(D, α) with $D \in F$ and D is satisfied by α .

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Let $C_1 = (\bar{x})$, so $\alpha_1 = x$.

The positive reduct $p(F, \alpha_1) = (\bar{x}) \wedge (x) \wedge (x)$ is **unsatisfiable**.

Let $C_2 = (\bar{x} \vee \bar{y})$, so $\alpha_2 = x y$.

The positive reduct $p(F, \alpha_2) = (\bar{x} \vee \bar{y}) \wedge (x \vee y) \wedge (x \vee \bar{y})$ is **satisfiable**.

Conditional Autarkies

Autarkies

A non-empty assignment α is an **autarky** for formula F if every clause $C \in F$ that is **touched** by α is also **satisfied** by α .

A **pure literal** and a **satisfying assignment** are autarkies.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Assignment $\alpha_1 = \bar{z}$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Assignment $\alpha_2 = x \bar{y} z$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

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Given an assignment α , $F|_{\alpha}$ denotes a formula F without the clauses satisfied by α and without the literals falsified by α .

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula F .

Then, F and $F|_{\alpha}$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a **conditional autarky** for formula F if α_{aut} is an autarky for $F|_{\alpha_{\text{con}}}$.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \bar{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\bar{y}$ is a conditional autarky for F :

$$\alpha_{\text{aut}} = \bar{y} \text{ is an autarky for } F|_{\alpha_{\text{con}}} = (\bar{y} \vee \bar{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a **conditional autarky** for formula F .
Then F and $F \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\bar{x} \vee \bar{y})$.

Learning PR clauses

Theorem

Given a formula F and an assignment α . Every satisfying assignment ω of $p(F, \alpha)$ is a conditional autarky of F .

Recall: Given a formula F and a clause C . Let α denote the smallest assignment that falsifies C . C is SET w.r.t. F if and only if $p(F, \alpha)$ is **satisfiable**.

Let assignment ω satisfy $p(F, \alpha)$. Removing all but one of the literals in C that are satisfied by ω results in a **PR clause** w.r.t. F .

The Algorithm

Pseudo-Code of CDCL (formula F)

```
1    $\alpha := \emptyset$ 
2   forever do
3      $\alpha := \text{Simplify}(F, \alpha)$ 
4     if  $F|_{\alpha}$  contains a falsified clause then
5        $C := \text{AnalyzeConflict}()$ 
6       if  $C$  is the empty clause then return unsatisfiable
7        $F := F \cup \{C\}$ 
8        $\alpha := \text{BackJump}(C, \alpha)$ 
13  else
14     $I := \text{Decide}()$ 
15    if  $I$  is undefined then return satisfiable
16     $\alpha := \alpha \cup \{I\}$ 
```

Pseudo-Code of SDCL (formula F)

```
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6       if  $C$  is the empty clause then return unsatisfiable
7        $F := F \cup \{C\}$ 
8        $\alpha := \text{BackJump}(C, \alpha)$ 
9     else if  $p(F, \alpha)$  is satisfiable then
10       $C := \text{AnalyzeWitness}()$ 
11       $F := F \cup \{C\}$ 
12       $\alpha := \text{BackJump}(C, \alpha)$ 
13    else
14       $I := \text{Decide}()$ 
15      if  $I$  is undefined then return satisfiable
16       $\alpha := \alpha \cup \{I\}$ 
```

Evaluation

Benchmark Suite: Pigeon Hole Formulas

Can $n+1$ pigeons be placed in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \leq p \leq n+1} (x_{1,p} \vee \dots \vee x_{n,p}) \wedge \bigwedge_{1 \leq h \leq n} \bigwedge_{1 \leq p < q \leq n+1} (\bar{x}_{h,p} \vee \bar{x}_{h,q})$$

The binary clauses encode the constraint $\leq_1 (x_{h,1}; \dots; x_{h,n+1})$.

There exists **more compact encodings**, such as the sequential counter and minimal encoding, for at-most-one constraints.

We include these encodings to evaluate the robustness of the solver.

Tool Comparison

We used three tools in our evaluation:

- **EBDDRES**: A tool based on binary decision diagrams that can convert a refutation into an extended resolution proof.
- **GLUCOSER**: A SAT solver with **extended learning**, i.e., a technique that introduces new variables and could potentially solve pigeon hole formulas in polynomial time.
- **LINGELING (PR)**: Our SDCL solver.

Results on Small Pigeon Hole Formulas

<i>formula</i>	input		EBDDRES		GLUCOSER		LINGELING (PR)	
	#var	#cls	time	#node	time	#lemma	time	#lemma
<i>PHP</i> ₁₀ -std	110	561	1.00	3M	22.71	329,470	0.07	329
<i>PHP</i> ₁₁ -std	132	738	3.47	9M	146.61	1,514,845	0.11	439
<i>PHP</i> ₁₂ -std	156	949	10.64	27M	307.29	2,660,358	0.16	571
<i>PHP</i> ₁₃ -std	182	1,197	30.81	76M	982.84	6,969,736	0.22	727
<i>PHP</i> ₁₀ -seq	220	311	OF	—	1.62	25,712	0.07	327
<i>PHP</i> ₁₁ -seq	264	375	OF	—	6.94	77,747	0.10	437
<i>PHP</i> ₁₂ -seq	312	445	OF	—	19.40	174,084	0.14	569
<i>PHP</i> ₁₃ -seq	364	521	OF	—	172.76	1,061,318	0.18	725
<i>PHP</i> ₁₀ -min	180	281	28.60	81M	0.64	15,777	0.06	329
<i>PHP</i> ₁₁ -min	220	342	143.92	399M	1.82	34,561	0.10	439
<i>PHP</i> ₁₂ -min	264	409	OF	—	9.87	121,321	0.13	571
<i>PHP</i> ₁₃ -min	312	482	OF	—	57.66	483,789	0.18	727

OF = 32-bit overflow

Results on Large Pigeon Hole Formulas

<i>formula</i>	input		EBDDRES		GLUCOSER		LINGELING (PR)	
	#var	#cls	time	#node	time	#lemma	time	#lemma
<i>PHP</i> ₂₀ -std	420	4,221	OF	—	TO	—	1.61	2,659
<i>PHP</i> ₃₀ -std	930	13,981	OF	—	TO	—	13.45	8,989
<i>PHP</i> ₄₀ -std	1,640	32,841	OF	—	TO	—	67.41	21,319
<i>PHP</i> ₅₀ -std	2,550	63,801	OF	—	TO	—	241.14	41,649
<i>PHP</i> ₂₀ -seq	840	1,221	OF	—	TO	—	1.05	2,657
<i>PHP</i> ₃₀ -seq	1,860	2,731	OF	—	TO	—	6.55	8,987
<i>PHP</i> ₄₀ -seq	3,280	4,841	OF	—	TO	—	27.10	21,317
<i>PHP</i> ₅₀ -seq	5,100	7,551	OF	—	TO	—	86.30	41,647
<i>PHP</i> ₂₀ -min	760	1,161	OF	—	TO	—	1.03	2,659
<i>PHP</i> ₃₀ -min	1,740	2,641	OF	—	TO	—	6.30	8,989
<i>PHP</i> ₄₀ -min	3,120	4,721	OF	—	TO	—	26.65	21,319
<i>PHP</i> ₅₀ -min	4,900	7,401	OF	—	TO	—	85.00	41,649

OF = 32-bit overflow

TO = timeout of 9000 seconds

Conclusions and Future Work

Conclusions

SDCL generalizes the well-known CDCL paradigm by allowing to prune branches that are potentially satisfiable:

- Such branches can be found using the **positive reduct**;
- Pruning can be expressed in the **PR proof system**;
- Runtime and proofs can be **exponentially** smaller.

Our SDCL solver finds short proofs of pigeon hole formulas:

- Integrated in the state-of-the-art solver **Lingeling**;
- Linear sized proofs in $\mathcal{O}(n^3)$ can be found fully automatically;
- The implementation is efficient, **robust**, and open source.

Future Work

- SDCL likely requires different heuristics compared to CDCL
- Can more branches be pruned using stronger SAT calls?
- How to minimize clauses from pruning through satisfaction?
- Can SLS techniques be used to find conditional autarkies?

PRuning Through Satisfaction

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Haifa Verification Conference

November 15, 2017