Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

THE UNIVERSITY OF
TEXAS
— AT AUSTIN —

JℲU
JOHANNES KEPLER
UNIVERSITÄT LINZ

Starting at Carnegie Mellon University in August

SAT 2019 Conference, Lisbon   July 9, 2019

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

$c_{1,1} = a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1}$

$c_{1,2} = a_{1,1} \cdot b_{1,2} + a_{1,2} \cdot b_{2,2}$

$c_{2,1} = a_{2,1} \cdot b_{1,1} + a_{2,2} \cdot b_{2,1}$

$c_{2,2} = a_{2,1} \cdot b_{1,2} + a_{2,2} \cdot b_{2,2}$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

$c_{1,1} = M_1 + M_4 - M_5 + M_7$

$c_{1,2} = M_3 + M_5$

$c_{2,1} = M_2 + M_4$

$c_{2,2} = M_1 - M_2 + M_3 + M_6$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

. . . where

$$M_1 = (a_{1,1} + a_{2,2}) \cdot (b_{1,1} + b_{2,2})$$
$$M_2 = (a_{2,1} + a_{2,2}) \cdot b_{1,1}$$
$$M_3 = a_{1,1} \cdot (b_{1,2} - b_{2,2})$$
$$M_4 = a_{2,2} \cdot (b_{2,1} - b_{1,1})$$
$$M_5 = (a_{1,1} + a_{1,2}) \cdot b_{2,2}$$
$$M_6 = (a_{2,1} - a_{1,1}) \cdot (b_{1,1} + b_{1,2})$$
$$M_7 = (a_{1,2} - a_{2,2}) \cdot (b_{2,1} + b_{2,2})$$

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $\mathcal{O}(n^{\omega})$ operations in the ground domain.

Matrix Multiplication: Introduction

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $\mathcal{O}(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $\mathcal{O}(n^{\omega})$ operations in the ground domain.
- Then $2 \le \omega < 3$. What is the exact value?

Efficient Matrix Multiplication: Theory

- Strassen 1969: $\omega \leq \log_2 7 \leq 2.807$

Efficient Matrix Multiplication: Theory

- Strassen 1969: $\omega \leq \log_2 7 \leq 2.807$
- Pan 1978: $\omega \leq 2.796$
- Bini et al. 1979: $\omega \leq 2.7799$
- Schönhage 1981: $\omega \leq 2.522$
- Romani 1982: $\omega \leq 2.517$
- Coppersmith/Winograd 1981: $\omega \leq 2.496$
- Strassen 1986: $\omega \leq 2.479$
- Coppersmith/Winograd 1990: $\omega \leq 2.376$

Efficient Matrix Multiplication: Theory

- Strassen 1969: $\omega \leq \log_2 7 \leq 2.807$
- Pan 1978: $\omega \leq 2.796$
- Bini et al. 1979: $\omega \leq 2.7799$
- Schönhage 1981: $\omega \leq 2.522$
- Romani 1982: $\omega \leq 2.517$
- Coppersmith/Winograd 1981: $\omega \leq 2.496$
- Strassen 1986: $\omega \leq 2.479$
- Coppersmith/Winograd 1990: $\omega \leq 2.376$
- Stothers 2010: $\omega \leq 2.374$
- Williams 2011: $\omega \leq 2.3728642$
- Le Gall 2014: $\omega \leq 2.3728639$

Efficient Matrix Multiplication: Practice

- ▶ Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.

- Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.

Efficient Matrix Multiplication: Practice

- ▸ Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

- ▸ Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.

- ▸ Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.

- ▸ Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

Efficient Matrix Multiplication: Practice

- Only Strassen's algorithm beats the classical algorithm for reasonable problem sizes.

- Want: a matrix multiplication algorithm that beats Strassen's algorithm for matrices of moderate size.

- Idea: instead of dividing the matrices into 2×2-block matrices, divide them into 3×3-block matrices.

- Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- Answer: Nobody knows.

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)

The 3x3 Case is Still Open

Question: What's the minimal number of multiplications needed to multiply two 3×3 matrices?

- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)
- maximal number of multiplications allowed if we want to beat Strassen: 21 (because $\log_3 21 < \log_2 7 < \log_3 22$).

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

$c_{1,1} = -M_6 + M_{14} + M_{19}$

$c_{2,1} = M_2 + M_3 + M_4 + M_6 + M_{14} + M_{16} + M_{17}$

$c_{3,1} = M_6 + M_7 - M_8 + M_{11} + M_{12} + M_{13} - M_{14}$

$c_{1,2} = M_1 - M_4 + M_5 - M_6 - M_{12} + M_{14} + M_{15}$

$c_{2,2} = M_2 + M_4 - M_5 + M_6 + M_{20}$

$c_{3,2} = M_{12} + M_{13} - M_{14} - M_{15} + M_{22}$

$c_{1,3} = -M_6 - M_7 + M_9 + M_{10} + M_{14} + M_{16} + M_{18}$

$c_{2,3} = M_{14} + M_{16} + M_{17} + M_{18} + M_{21}$

$c_{3,3} = M_6 + M_7 - M_8 - M_9 + M_{23}$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where . . .

$$M_1 = (-a_{1,1} + a_{1,2} + a_{1,3} - a_{2,1} + a_{2,2} + a_{3,2} + a_{3,3}) \cdot b_{2,2}$$
$$M_2 = (a_{1,1} + a_{2,1}) \cdot (b_{1,2} + b_{2,2})$$
$$M_3 = a_{2,2} \cdot (b_{1,1} - b_{1,2} + b_{2,1} - b_{2,2} - b_{2,3} + b_{3,1} - b_{3,3})$$
$$M_4 = (-a_{1,1} - a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2} + b_{2,2})$$
$$M_5 = (-a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2})$$
$$M_6 = -a_{1,1} \cdot b_{1,1}$$
$$M_7 = (a_{1,1} + a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3} + b_{2,3})$$
$$M_8 = (a_{1,1} + a_{3,1}) \cdot (-b_{1,3} + b_{2,3})$$
$$M_9 = (a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3})$$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where ...

$$M_{10} = (a_{1,1} + a_{1,2} - a_{1,3} - a_{2,2} + a_{2,3} + a_{3,1} + a_{3,2}) \cdot b_{2,3}$$
$$M_{11} = (a_{3,2}) \cdot (-b_{1,1} + b_{1,3} + b_{2,1} - b_{2,2} - b_{2,3} - b_{3,1} + b_{3,2})$$
$$M_{12} = (a_{1,3} + a_{3,2} + a_{3,3}) \cdot (b_{2,2} + b_{3,1} - b_{3,2})$$
$$M_{13} = (a_{1,3} + a_{3,3}) \cdot (-b_{2,2} + b_{3,2})$$
$$M_{14} = a_{1,3} \cdot b_{3,1}$$
$$M_{15} = (-a_{3,2} - a_{3,3}) \cdot (-b_{3,1} + b_{3,2})$$
$$M_{16} = (a_{1,3} + a_{2,2} - a_{2,3}) \cdot (b_{2,3} - b_{3,1} + b_{3,3})$$
$$M_{17} = (-a_{1,3} + a_{2,3}) \cdot (b_{2,3} + b_{3,3})$$
$$M_{18} = (a_{2,2} - a_{2,3}) \cdot (b_{3,1} - b_{3,3})$$

Laderman's scheme from 1976

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

where ...

$$M_{19} = a_{1,2} \cdot b_{2,1}$$
$$M_{20} = a_{2,3} \cdot b_{3,2}$$
$$M_{21} = a_{2,1} \cdot b_{1,3}$$
$$M_{22} = a_{3,1} \cdot b_{1,2}$$
$$M_{23} = a_{3,3} \cdot b_{3,3}$$

Other schemes with 23 multiplications

- ▶ While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.

- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.

- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.

- Using altogether about 35 years of computation time, we found more than 13000 new schemes for 3×3 and 23, and we expect that there are many others.

Other schemes with 23 multiplications

- While Strassen's scheme is essentially the only way to do the 2×2 case with 7 multiplications, there are several distinct schemes for 3×3 matrices using 23 multiplications.

- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.

- Using altogether about 35 years of computation time, we found more than 13000 new schemes for 3×3 and 23, and we expect that there are many others.

- Unfortunately we found no scheme with only 22 multiplications

$$M_1 = (\alpha_{1,1}^{(1)} a_{1,1} + \alpha_{1,2}^{(1)} a_{1,2} + \cdots)(\beta_{1,1}^{(1)} b_{1,1} + \cdots)$$

$$M_2 = (\alpha_{1,1}^{(2)} a_{1,1} + \alpha_{1,2}^{(2)} a_{1,2} + \cdots)(\beta_{1,1}^{(2)} b_{1,1} + \cdots)$$

$$\vdots$$

$$c_{1,1} = \gamma_{1,1}^{(1)} M_1 + \gamma_{1,1}^{(2)} M_2 + \cdots$$

$$\vdots$$

Set $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for all i, j and compare coefficients.

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

The $\delta_{u,v}$ on the right refer to the Kronecker-delta, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise.

$3^6 = 729$ cubic equations

$23 \cdot 9 \cdot 3 = 621$ variables

How to Search for a Matrix Multiplication Scheme? (2)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

The $\delta_{u,v}$ on the right refer to the Kronecker-delta, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise.

$3^6 = 729$ cubic equations

$23 \cdot 9 \cdot 3 = 621$ variables

Laderman claims that he solved this system by hand, but he doesn't say exactly how.

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

The search space of the 3×3 case is enormous, even if $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ are restricted to the values in $\{-1, 0, 1\}$

How to Search for a Matrix Multiplication Scheme? (3)

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

The search space of the 3×3 case is enormous, even if $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ are restricted to the values in $\{-1, 0, 1\}$

Solution: Solve this system in $\mathbb{Z}_2$.

Reading $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables and $+$ as XOR, the problem becomes a SAT problem.

Notice that solutions in $\mathbb{Z}_2$ may not be solutions in $\mathbb{Z}$

Lifting

Remember the Brent equations:

$$\forall\, i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

- Suppose we know a solution in $\mathbb{Z}_2$.
- Assume it came from a solution in $\mathbb{Z}$ with coefficients in $\{-1,0,+1\}$.
- Then each $0 \in \mathbb{Z}_2$ was $0 \in \mathbb{Z}$ and each $1 \in \mathbb{Z}_2$ was $-1 \in \mathbb{Z}$ or $+1 \in \mathbb{Z}$.
- Plug the 0s of the $\mathbb{Z}_2$-solution into the Brent equations.
- Solve the resulting equations.

Lifting

Remember the Brent equations:

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

- Suppose we know a solution in $\mathbb{Z}_2$.
- Assume it came from a solution in $\mathbb{Z}$ with coefficients in $\{-1, 0, +1\}$.
- Then each $0 \in \mathbb{Z}_2$ was $0 \in \mathbb{Z}$ and each $1 \in \mathbb{Z}_2$ was $-1 \in \mathbb{Z}$ or $+1 \in \mathbb{Z}$.
- Plug the 0s of the $\mathbb{Z}_2$-solution into the Brent equations.
- Solve the resulting equations.

Can every $\mathbb{Z}_2$-solution be lifted to a $\mathbb{Z}$-solution in this way?

- No, and we found some which don't admit a lifting.
- But they are very rare. In almost all cases, the lifting succeeds.

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.

This gives the Brent equations (for 3×3 with 23 multiplications)

$$\forall\, i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.

Important challenge: how to break the symmetries?

Most effective approach so far: sort the $\delta_{j,k} \delta_{i,m} \delta_{l,n} = 1$ terms

Neighborhood Search

Neighborhood Search Results

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.
- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.

- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.

- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.

So what?

- Okay, so there are many more matrix multiplication methods for 3×3 matrices with 23 coefficient multiplications than previously known.

- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.

- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.

- In particular, it remains open whether there is a multiplication method for 3×3 matrices with 22 coefficient multiplications. If you find one, let us know.

What's Next?

Scheme Database

Check out our website for browsing through
the schemes and families we found:

http://www.algebra.uni-linz.ac.at/research/matrix-multiplication/

Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

THE UNIVERSITY OF
TEXAS
— AT AUSTIN —

JꓘU
JOHANNES KEPLER
UNIVERSITÄT LINZ

Starting at Carnegie Mellon University in August

SAT 2019 Conference, Lisbon   July 9, 2019