Encoding Redundancy for Satisfaction-Driven Clause Learning

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The Problem

Although SAT solvers can often handle gigantic formulas,
they sometimes fail miserably on seemingly easy problems.

Outline

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SAT Solving in Practice: Gigantic Search Trees

- SAT: Given a propositional formula, is it satisfiable?
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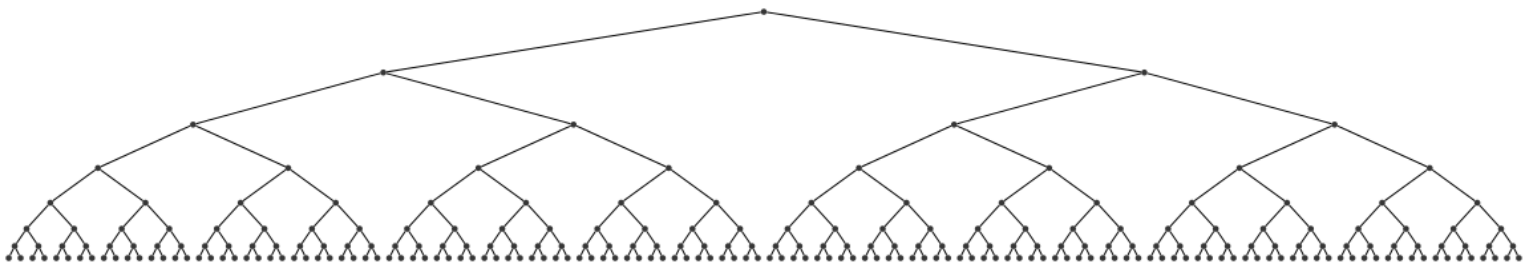

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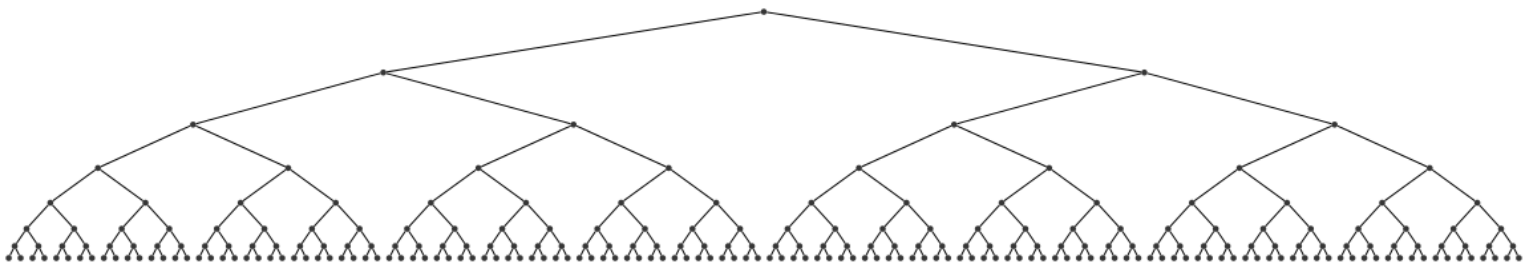

- Modern solvers often deal with millions of variables and clauses.

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- What if we encode it into SAT and pass it to a solver?

SAT Solver: 21 Pigeons Into 20 Holes?

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"*Arguably the single most studied combinatorial principle in all of proof complexity.*" [Nordström, SIGLOG News '15]

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  ➥ They need exponential time to solve these formulas.

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  2. a SAT solving paradigm harnessing the strength of PR: satisfaction-driven clause learning [Heule, K, Seidl, Biere; HVC '17].

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➥ Addition of redundant clauses can prune the search tree.

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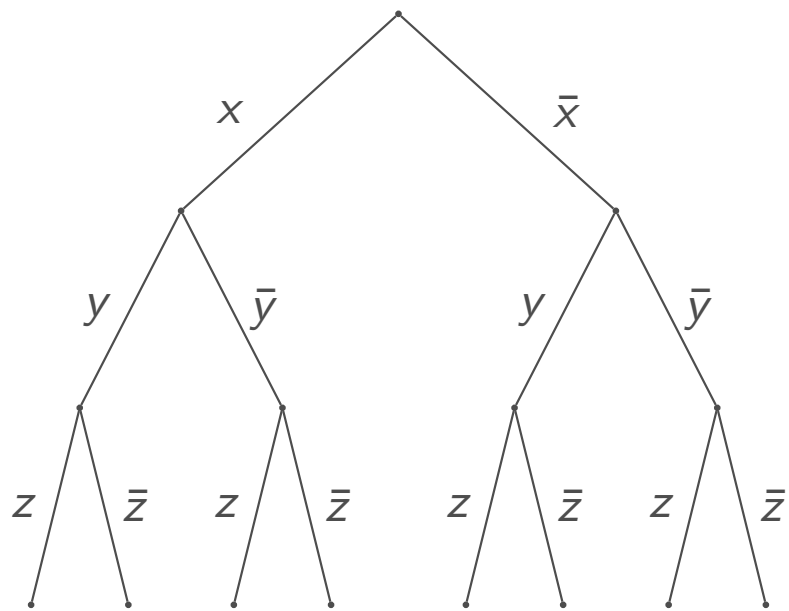

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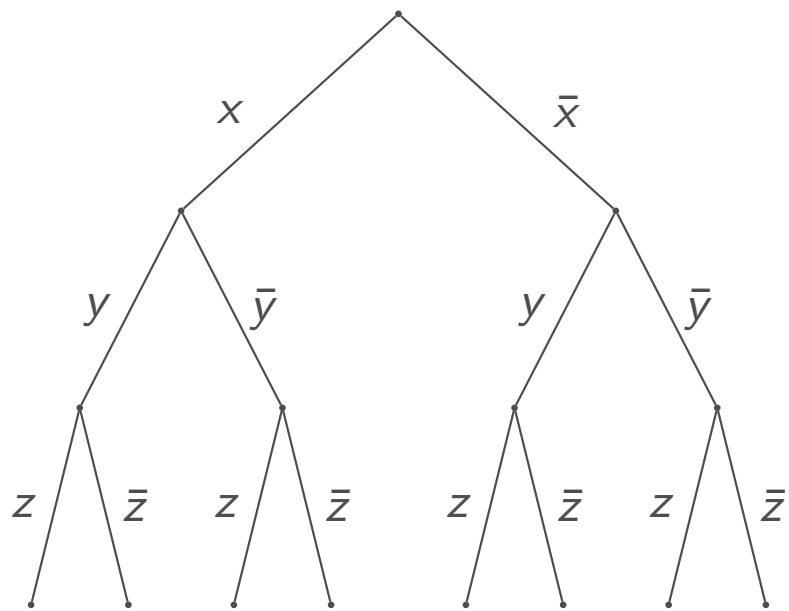

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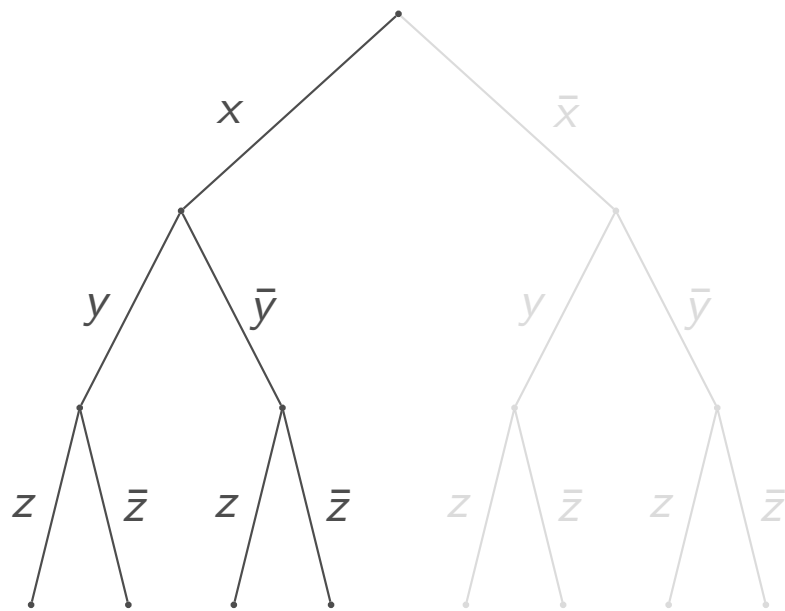

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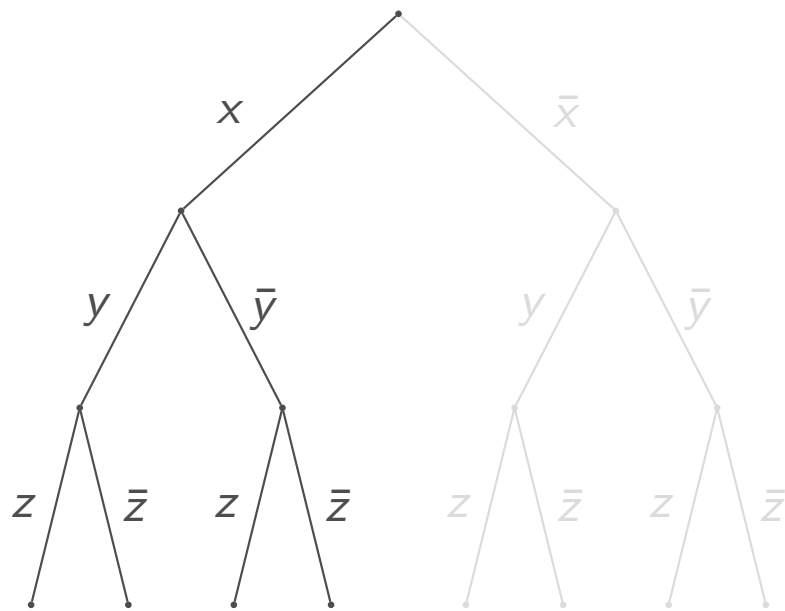

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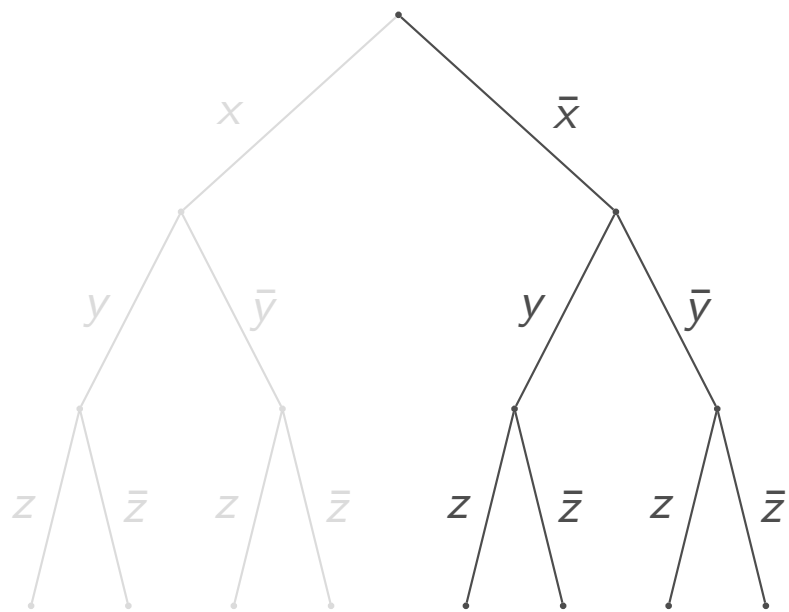

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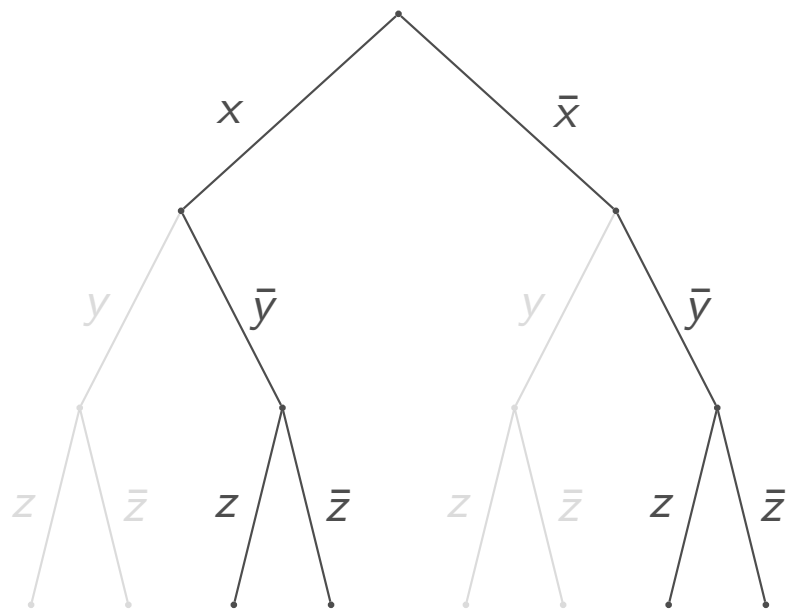

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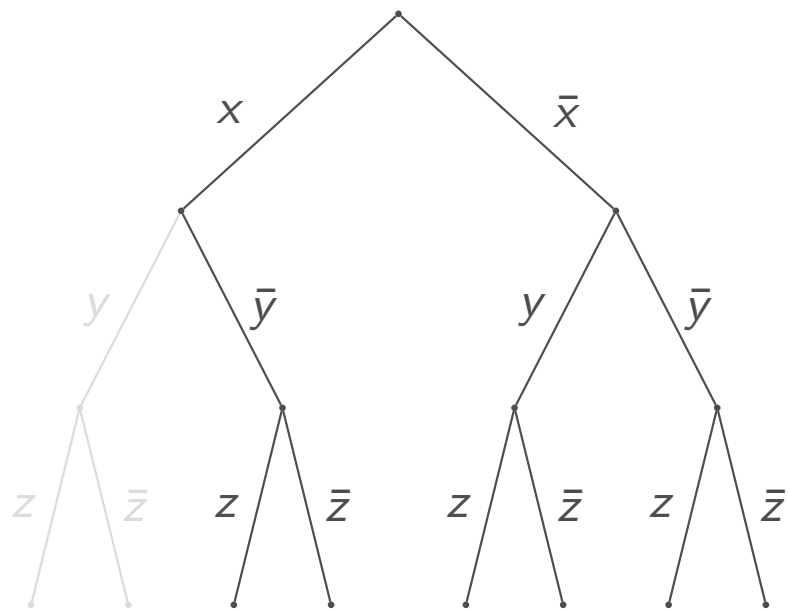

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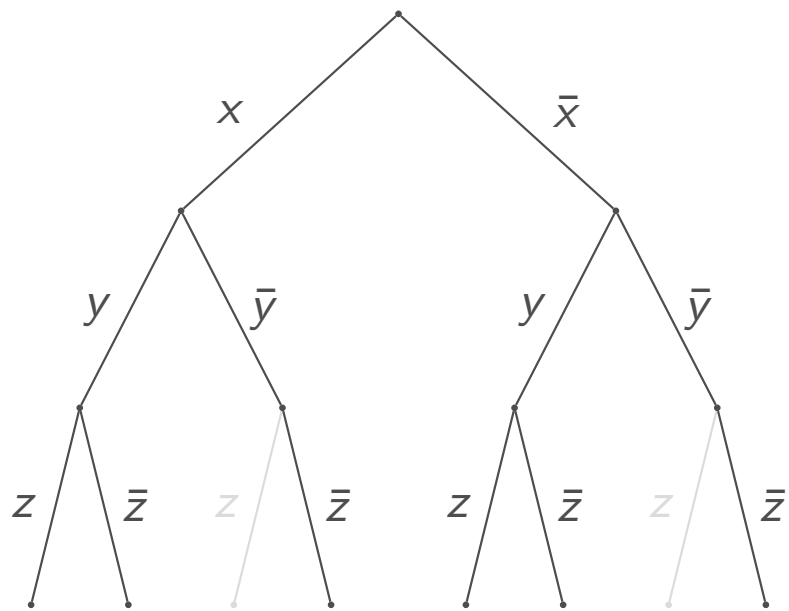

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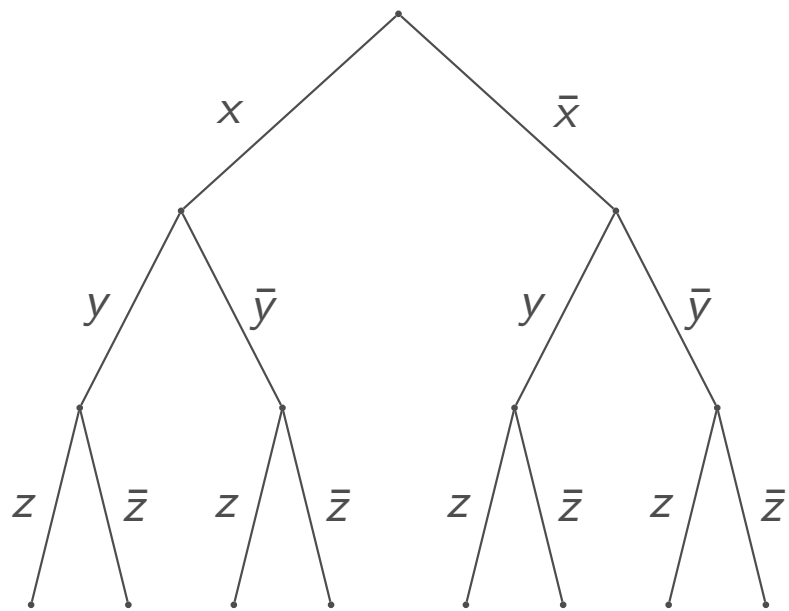

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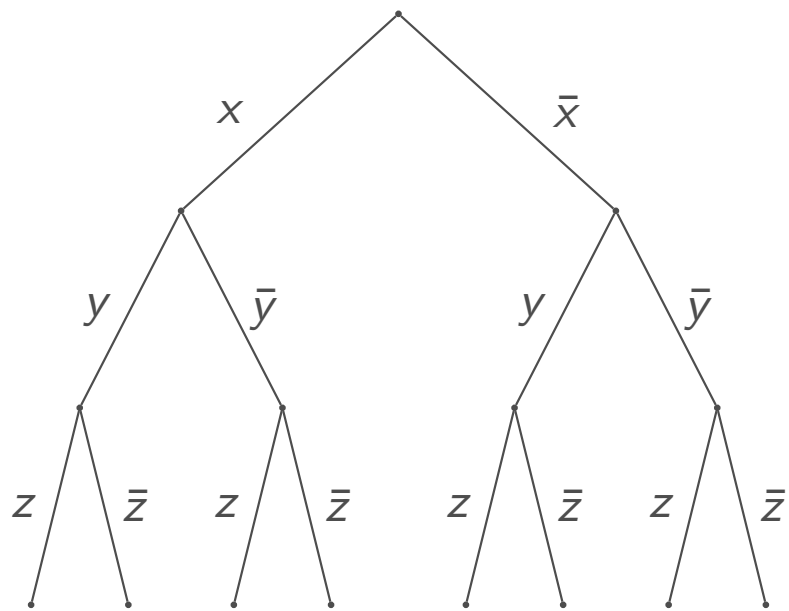

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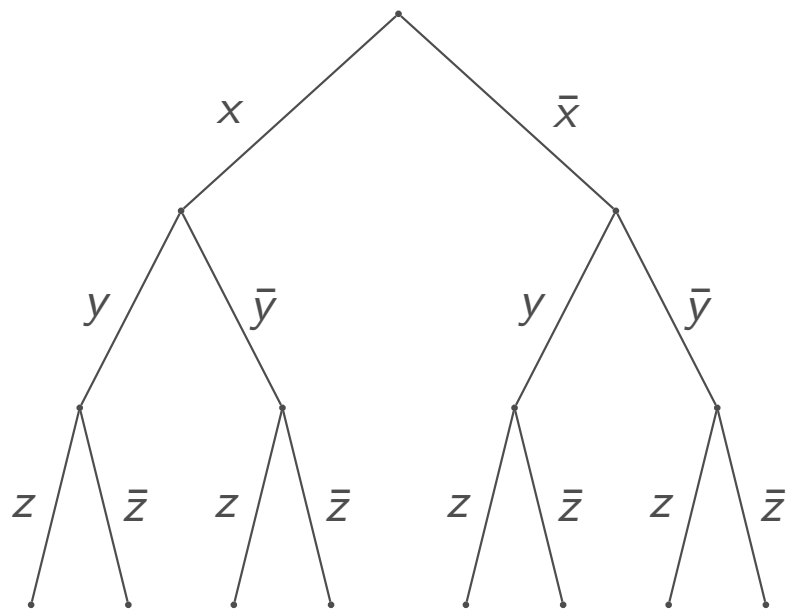

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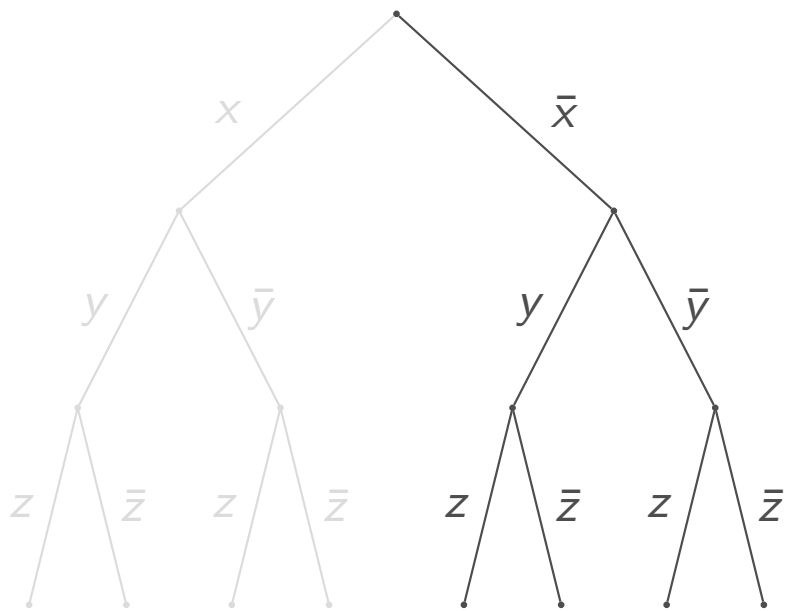

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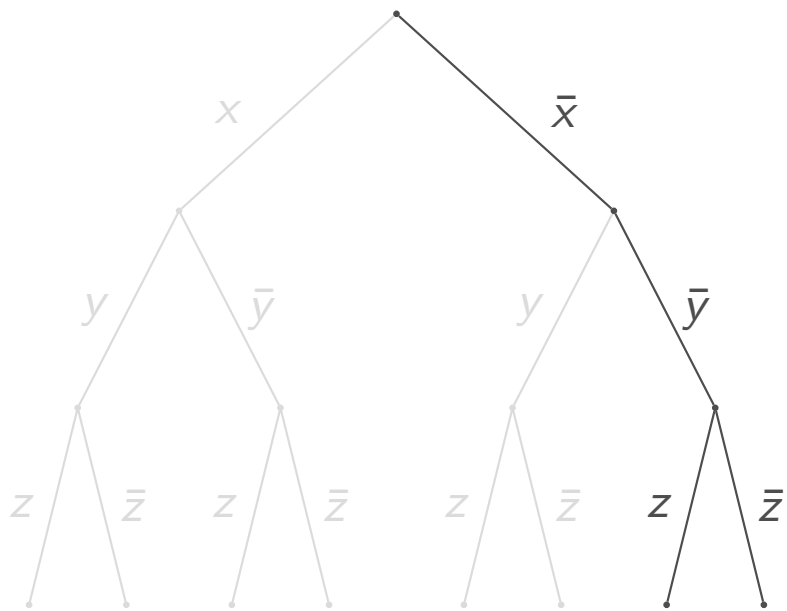

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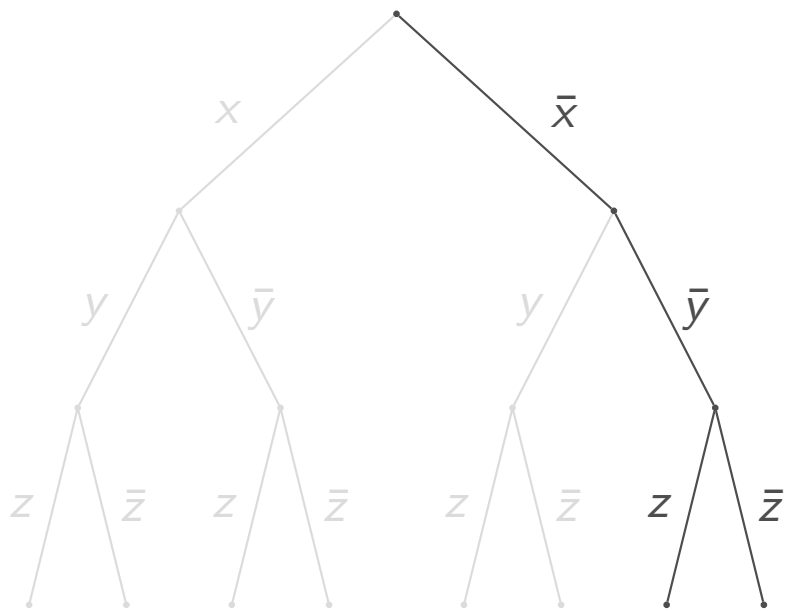

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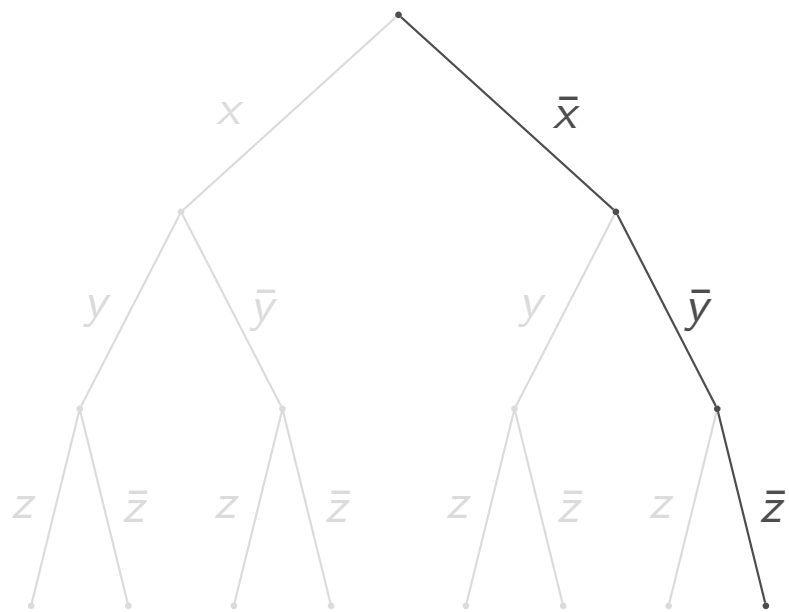

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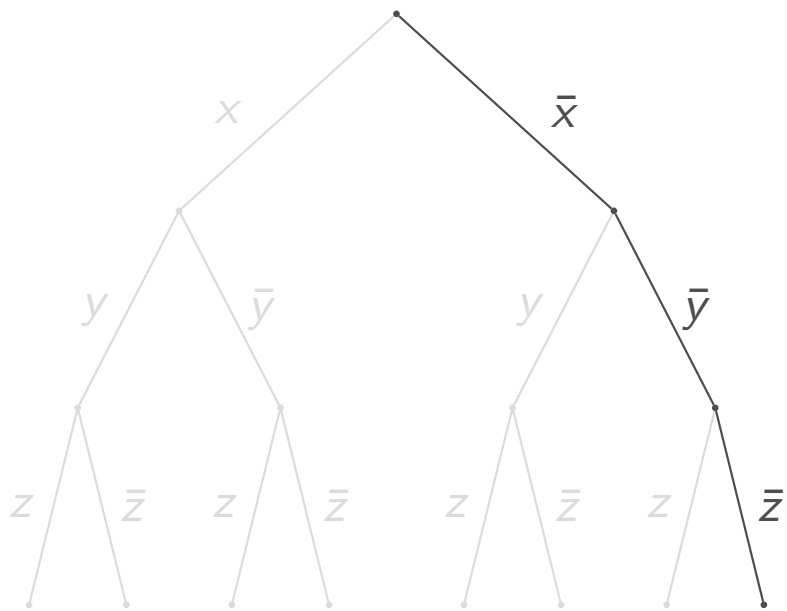

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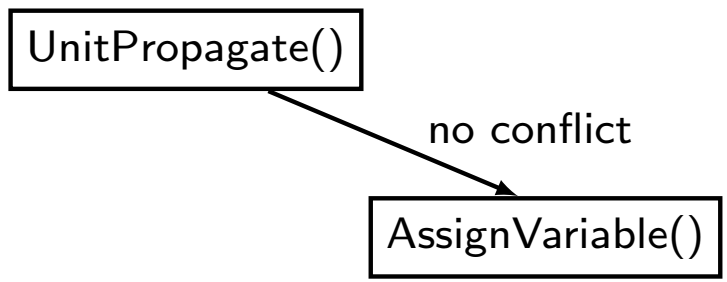

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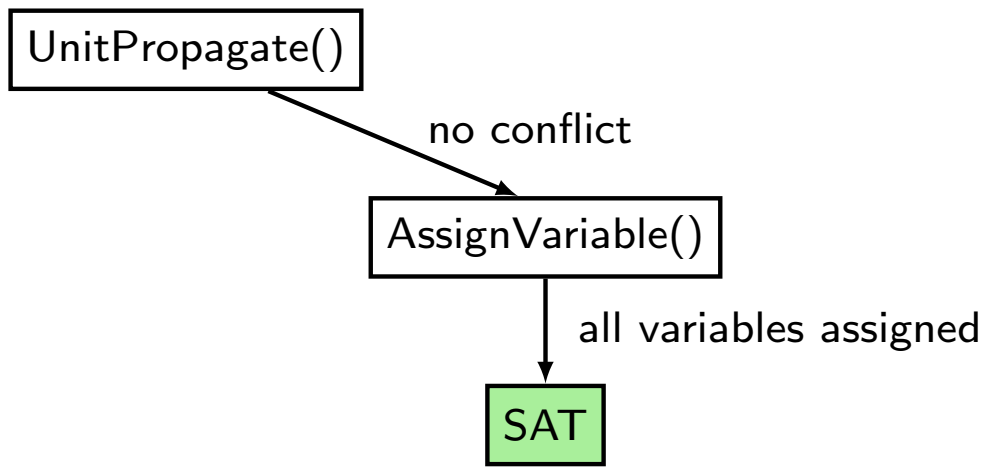

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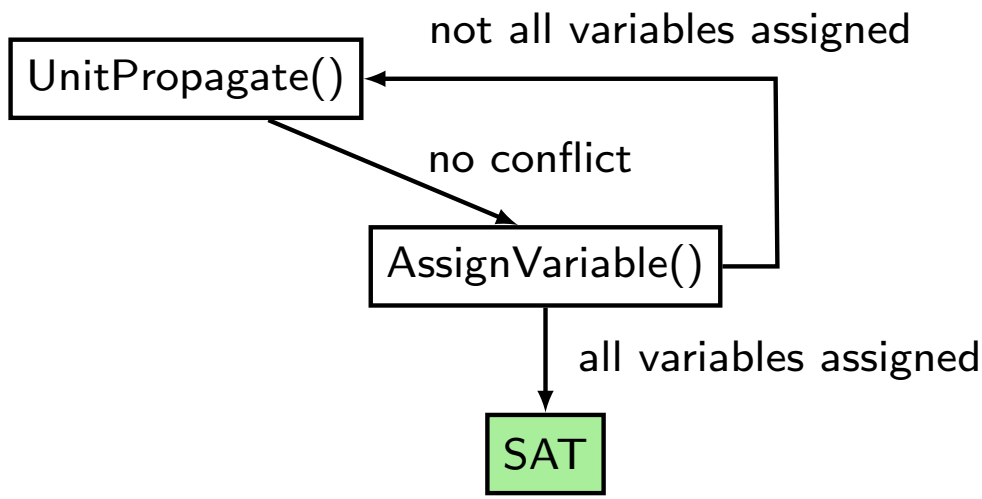

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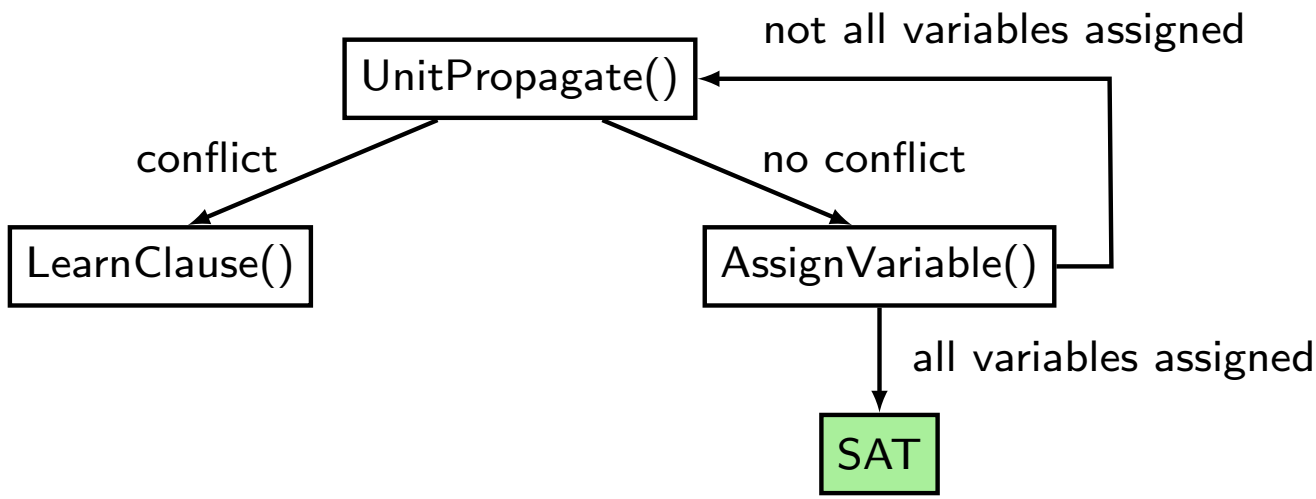

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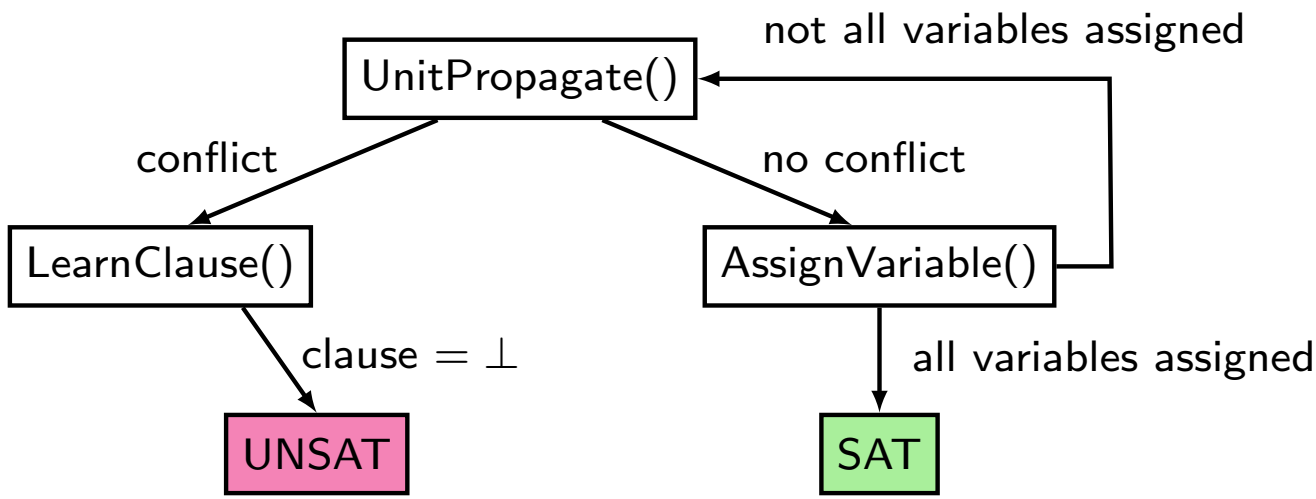

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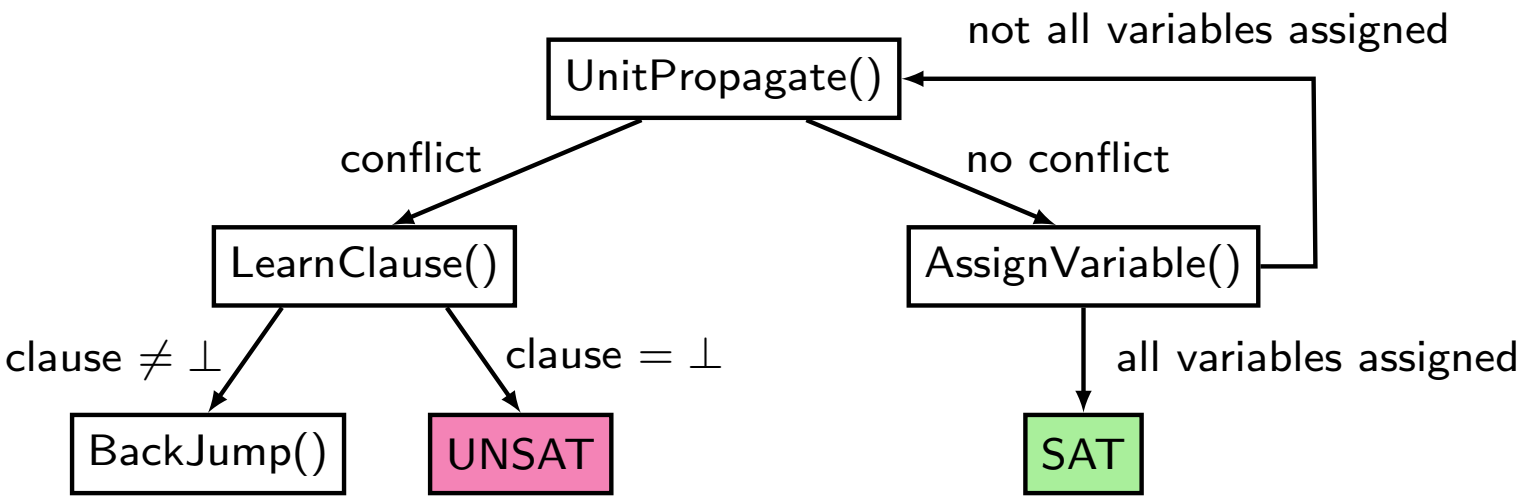

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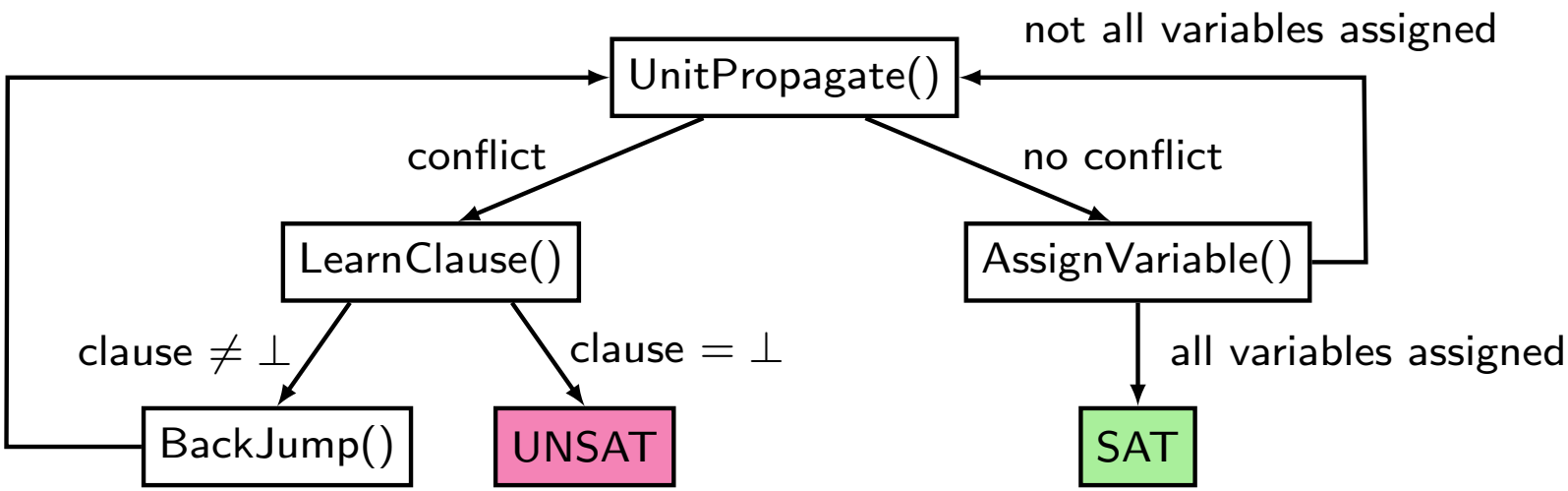

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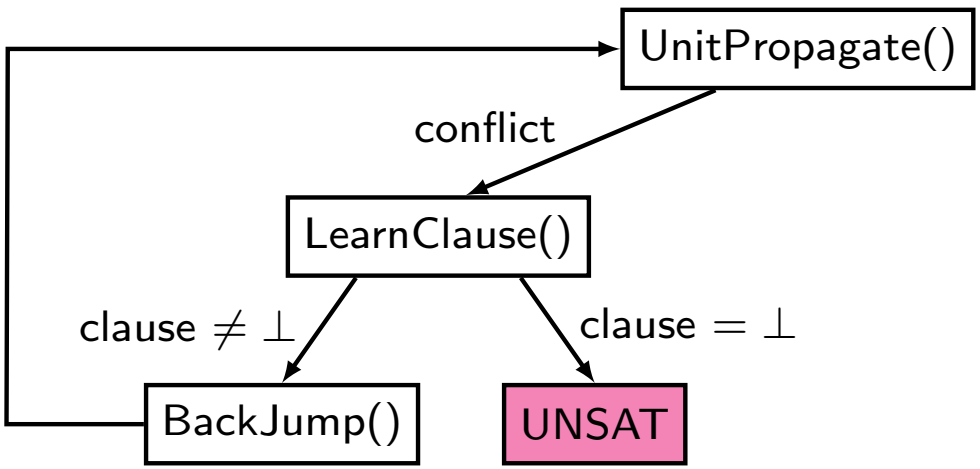

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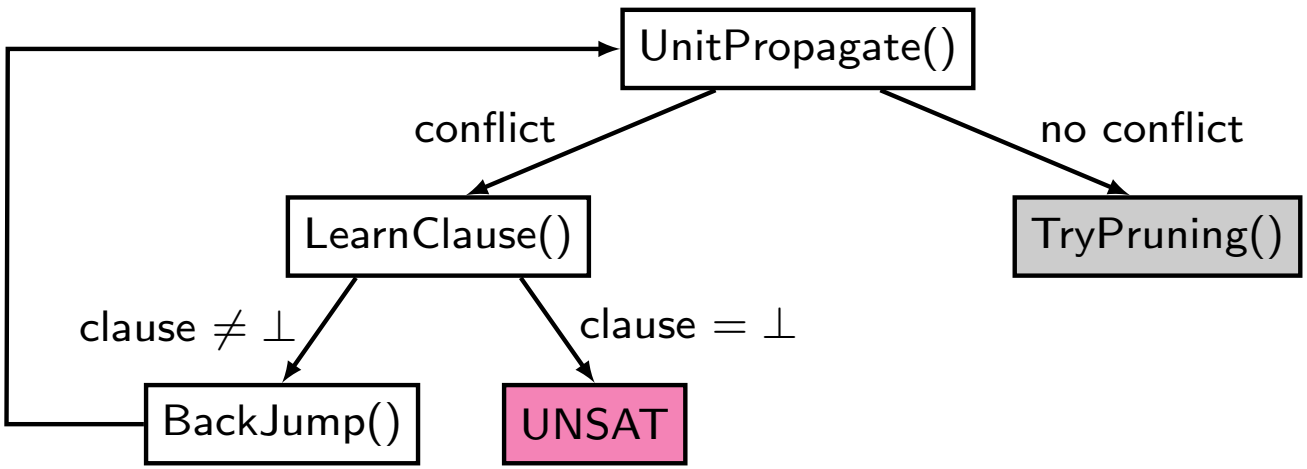

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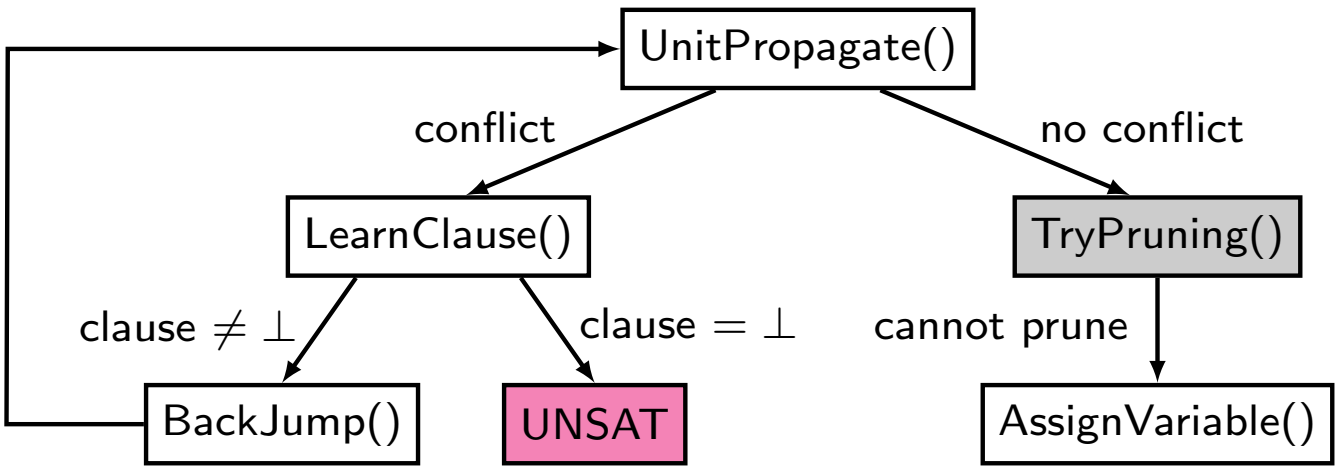

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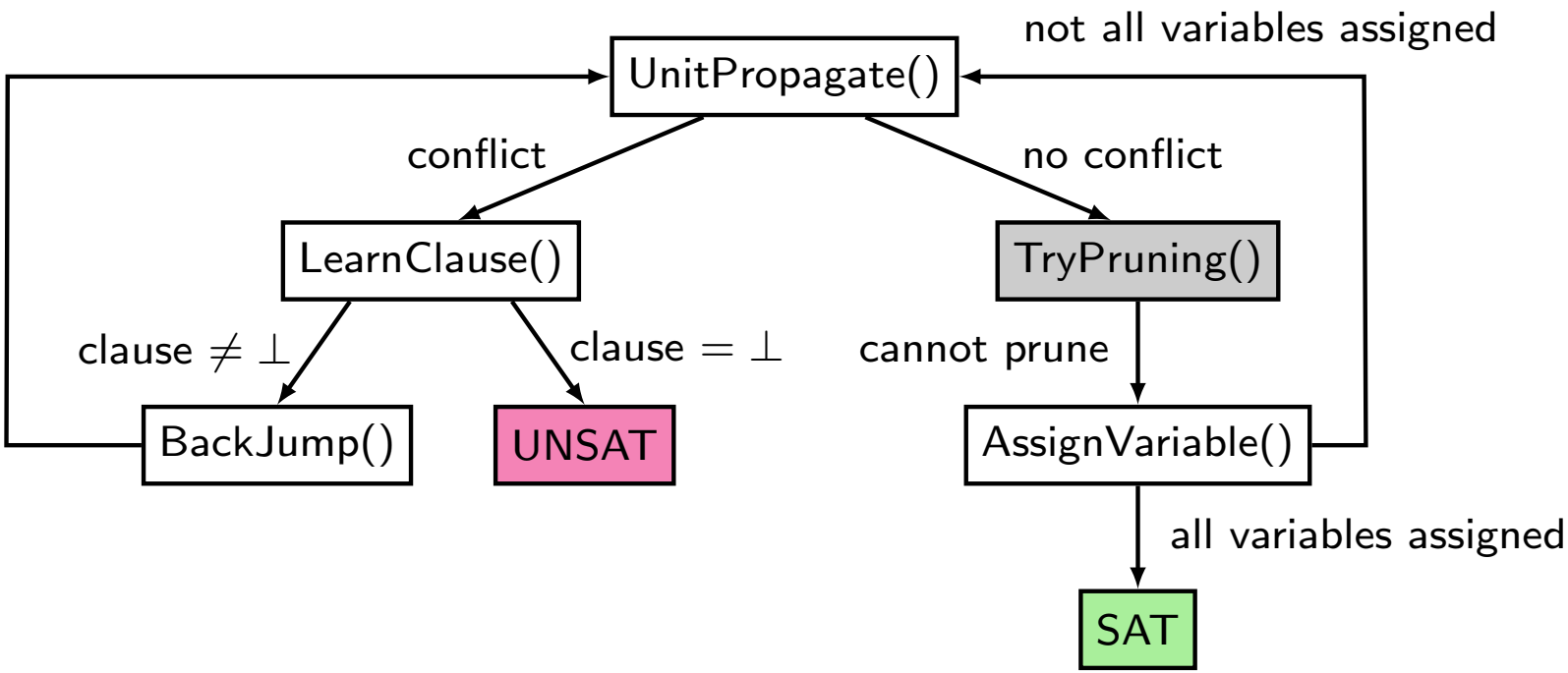

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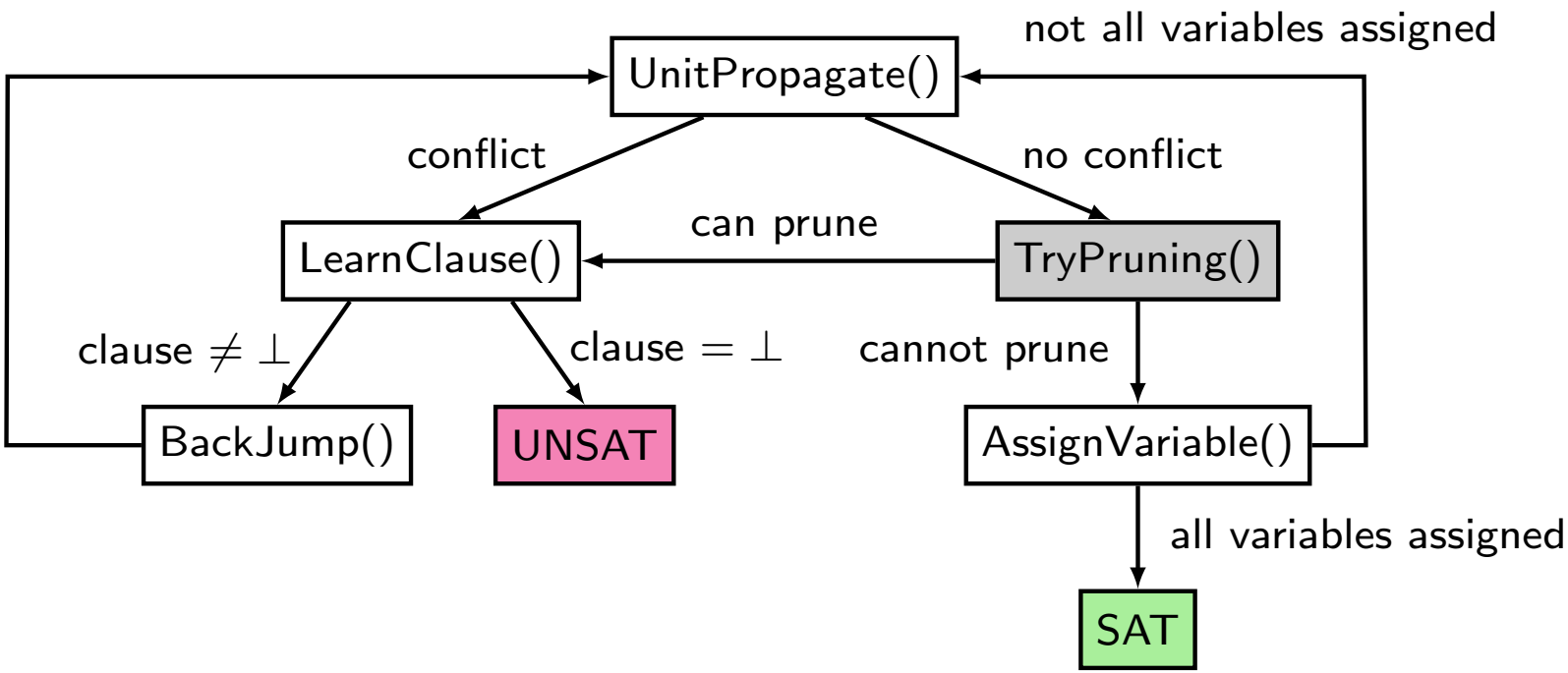

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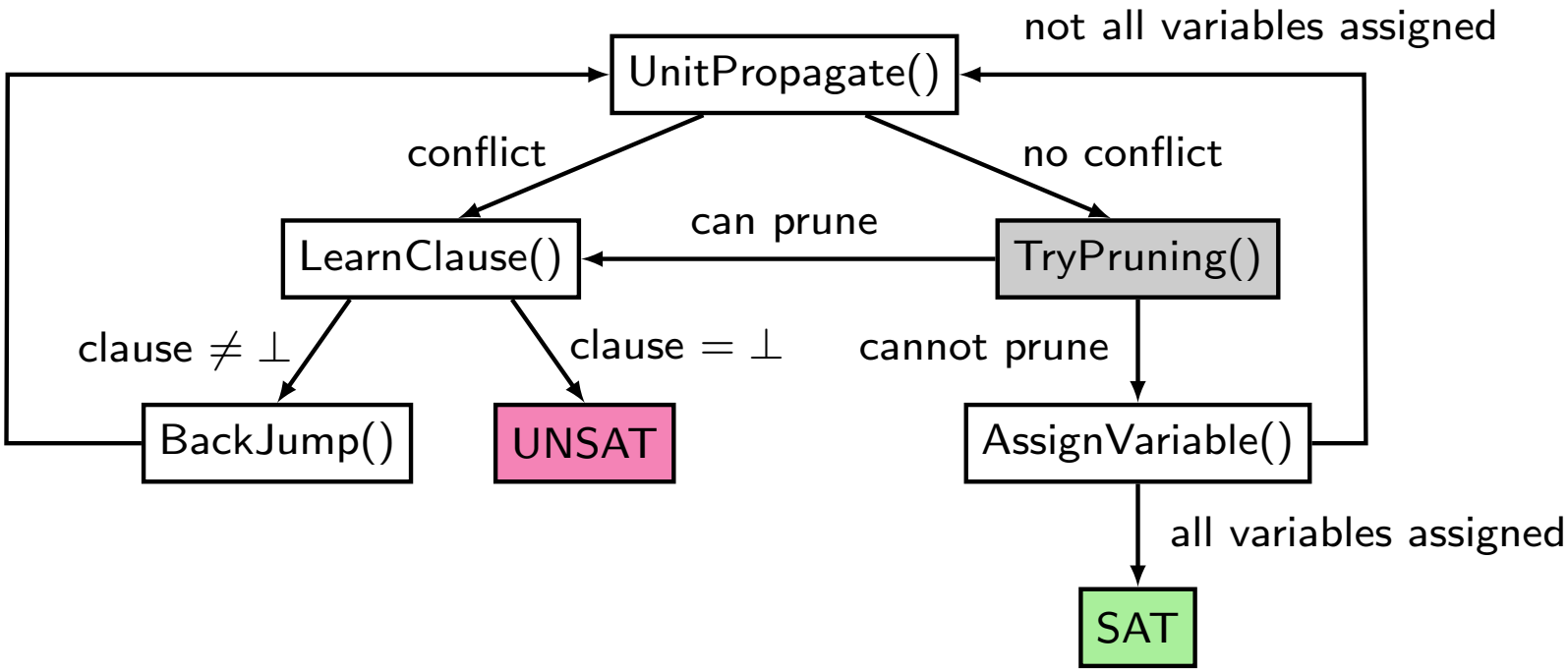

- Learned clauses are not necessarily implied (PR clauses).

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➥ Wanted: Better encodings for pruning!

Background

Contribution

Encodings for Stronger Pruning: Some Preliminaries

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- $\text{touched}_\alpha(C)$ denotes the subclause of C that is assigned by α.
- Notion of implication via unit propagation:
  - Clauses: $F \vdash_1 C$ iff unit propagation derives a conflict on $F \wedge \bar{C}$.
  - Formulas: $F \vdash_1 G$ iff $F \vdash_1 C$ for all $C \in G$.

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Let F be a formula and α an assignment. Then, the filtered positive reduct $\mathsf{f}_\alpha(F)$ of F and α is the formula $G \wedge \bar{\alpha}$ where $G = \{\mathsf{touched}_\alpha(D) \mid D \in F \text{ and } F|_\alpha \not\models_1 \mathsf{untouched}_\alpha(D)\}$.

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➥ Works very well in practice (see later)!

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  - positive reduct $\mathsf{p}_\alpha(F) = (x) \wedge (\bar{x}) \Rightarrow$ unsatisfiable $\Rightarrow$ can't prune α

Even Stronger Pruning: PR Reduct

- PR reduct (don't try to understand this):

Definition

Let F be a formula and α an assignment. Then, the PR reduct $\mathrm{pr}_\alpha(F)$ of F and α is the formula $G \wedge C$ where C is the clause that blocks α and G is the union of the following sets of clauses where all the s_i are new variables:

$$\{\bar{x}^p \vee \bar{x}^n \mid x \in \mathit{var}(F) \setminus \mathit{var}(\alpha)\},$$

$$\{\bar{s}_i \vee \mathrm{touched}_\alpha(D_i) \vee \mathrm{untouched}_\alpha(D_i)^p \mid D_i \in F\},$$

$$\{\overline{L^n} \vee s_i \mid D_i \in F \text{ and } L \subseteq \mathrm{untouched}_\alpha(D_i) \text{ such that } F|_\alpha \not\vdash_1 \mathrm{untouched}_\alpha(D_i) \setminus L\}.$$

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- Allows for even stronger pruning than the filtered positive reduct.
- Precisely characterizes propagation redundancy.
  - ➥ Extremely general redundancy notion (NP-hard).
- Has other nice theoretical properties.
- Doesn't work well in practice
  - ➥ Constructing and solving take too long.

Evaluation: SDCL in Practice

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- Robust w.r.t. scrambling for Tseitin formulas and mutilated chessboards.

Experimental Data: Pigeonhole Principle

Formula	MLBT	Plain	Pos. Red.	F. Red
hole20	> 3600	> 3600	0.26	0.49
hole30	> 3600	> 3600	1.96	4.03
hole40	> 3600	> 3600	9.02	19.54
hole50	> 3600	> 3600	28.63	65.90

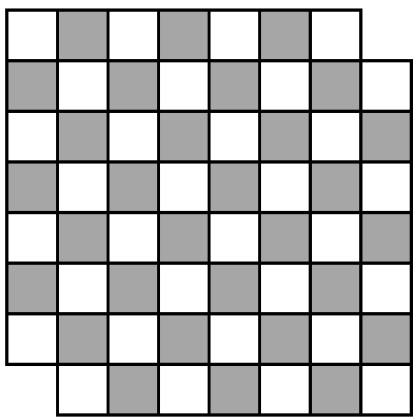
- MLBT – MapleLCMDistChronoBT (winner SAT Competition 2018)
- Plain – SaDiCaL in CDCL mode

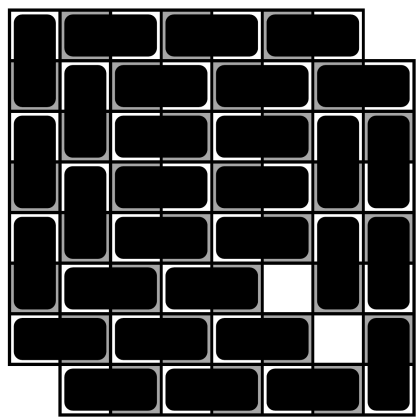
Experimental Data: Tseitin Formulas

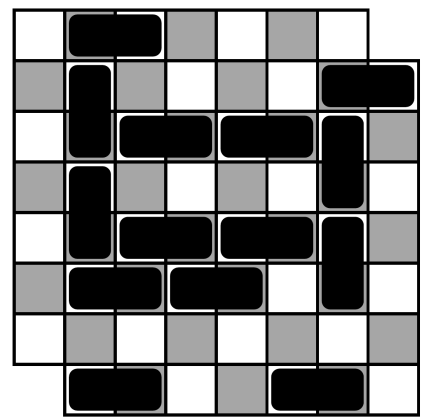
Formula	MLBT	Plain	Pos. Red.	F. Red
Urquhart-s3-b1	2.95	16.31	> 3600	0.02
Urquhart-s3-b2	1.36	2.82	> 3600	0.03
Urquhart-s3-b3	2.28	2.08	> 3600	0.03
Urquhart-s3-b4	10.74	7.65	> 3600	0.03
Urquhart-s4-b1	86.11	> 3600	> 3600	0.32
Urquhart-s4-b2	154.35	183.77	> 3600	0.11
Urquhart-s4-b3	258.46	129.27	> 3600	0.16
Urquhart-s4-b4	> 3600	> 3600	> 3600	0.14
Urquhart-s5-b1	> 3600	> 3600	> 3600	1.27
Urquhart-s5-b2	> 3600	> 3600	> 3600	0.58
Urquhart-s5-b3	> 3600	> 3600	> 3600	1.67
Urquhart-s5-b4	> 3600	> 3600	> 3600	2.91

Experimental Data: Mutilated Chessboards

Formula	MLBT	Plain	Pos. Red.	F. Red
mchess_15	51.53	2480.67	> 3600	13.14
mchess_16	380.45	2115.75	> 3600	15.52
mchess_17	2418.35	> 3600	> 3600	25.54
mchess_18	> 3600	> 3600	> 3600	43.88







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- Next step: SDCL for hard problems from cryptanalysis?