

A Little Blocked Literal Goes a Long Way

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UT Austin

JKU Linz



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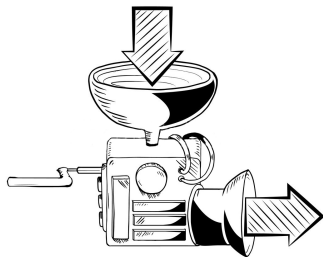
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- Brief overview of QBF and corresponding proof systems.
- **Main result: QRAT** simulates **long-distance resolution**.
 - QRAT is the QBF generalization of DRAT.
 - Simulation is **polynomial**.
- We have an **implementation** and **evaluation** of the simulation.

Satisfiability of Quantified Boolean Formulas (QSAT)

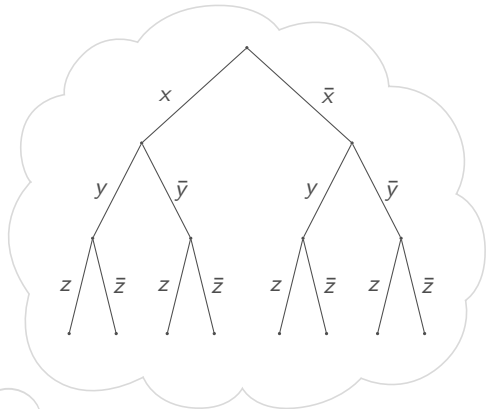
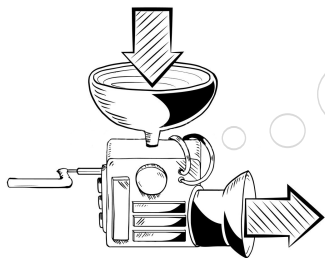
“For every truth value of x ,
does there exist a truth value of y ,
such that ...”

$$\forall x \exists y \forall z (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$$



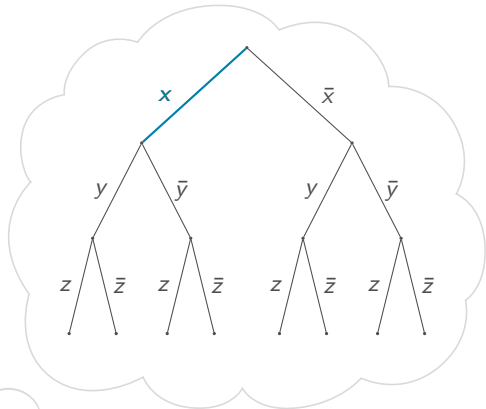
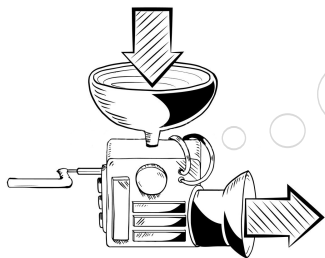
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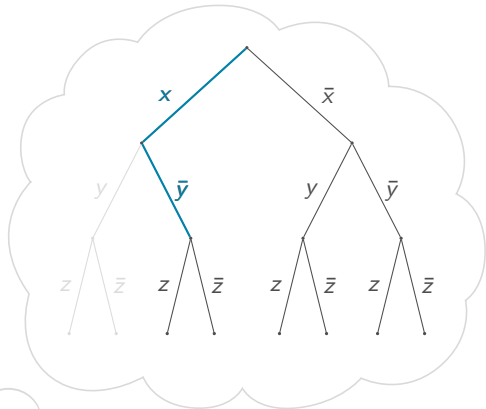
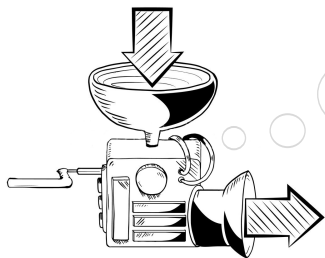
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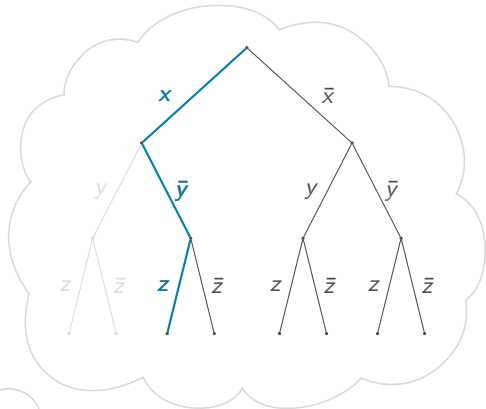
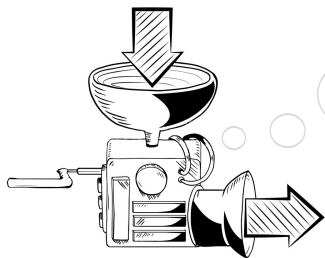
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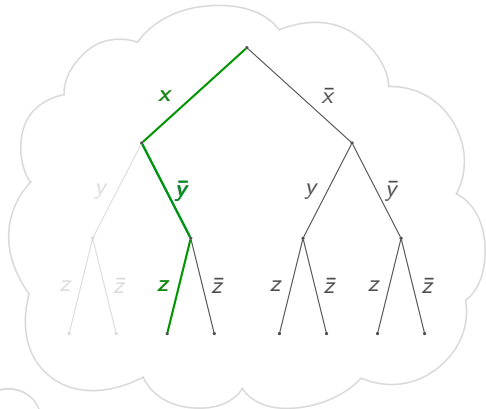
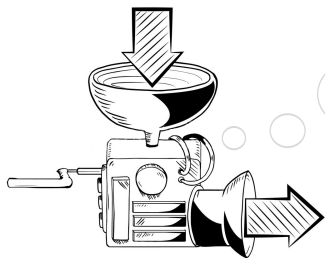
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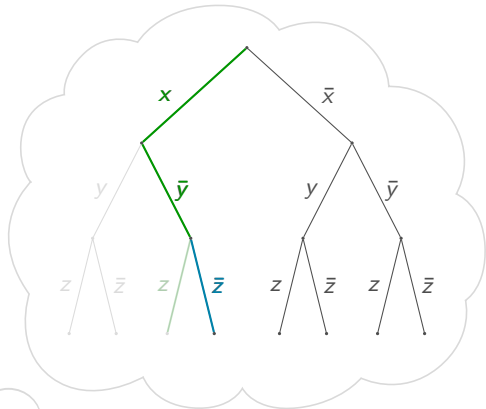
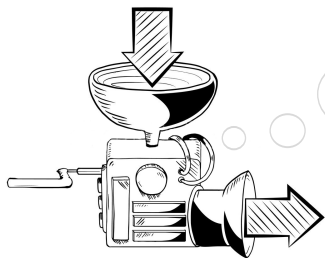
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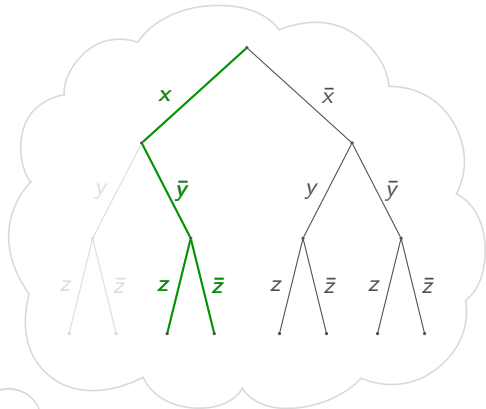
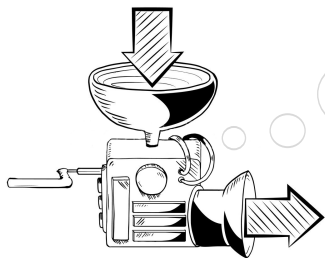
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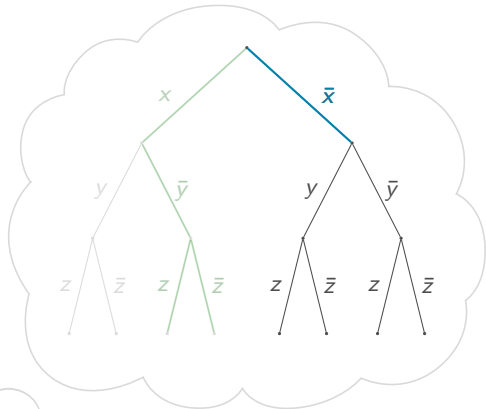
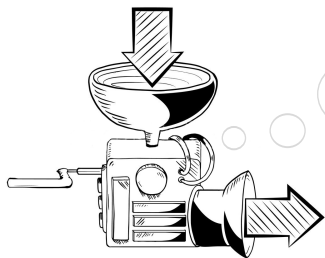
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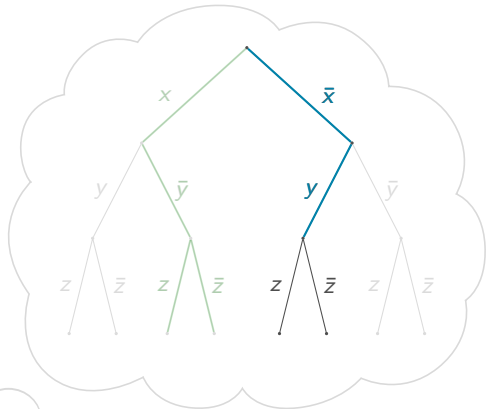
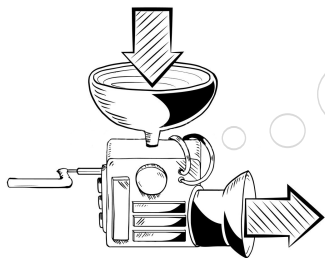
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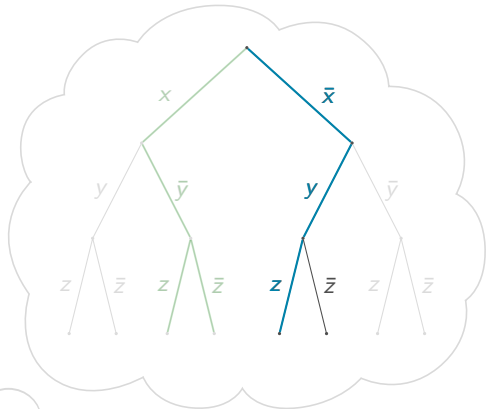
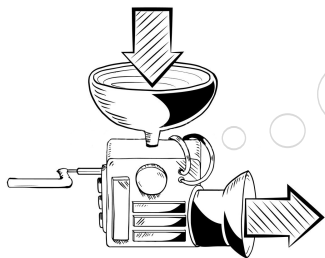
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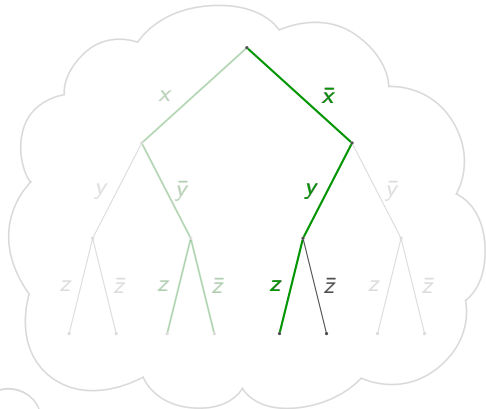
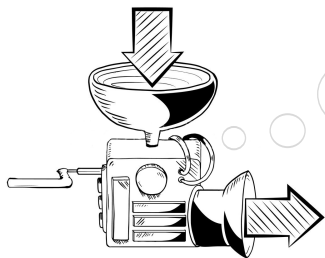
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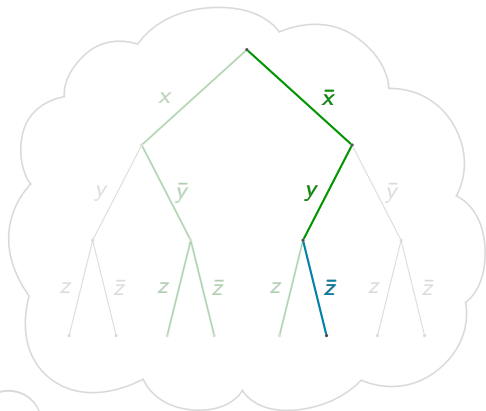
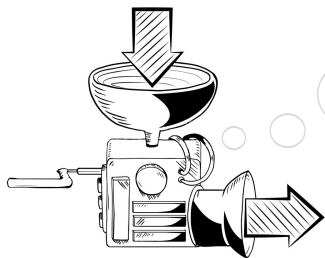
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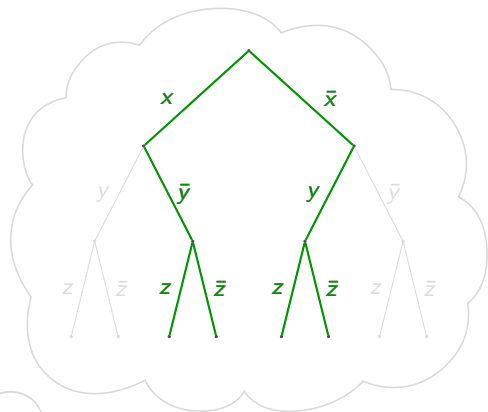
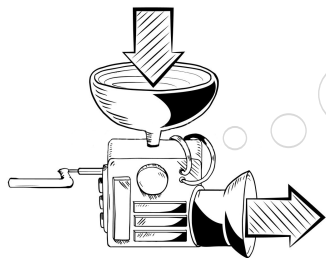
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Satisfiable

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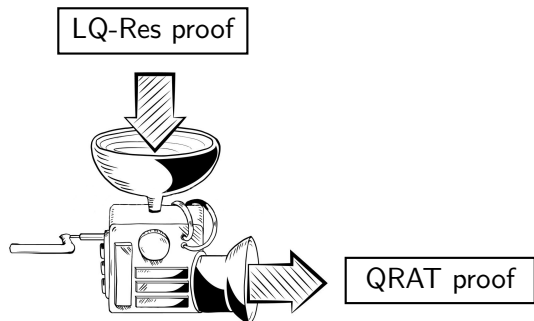
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- **Open question**: If there is a **short LQ-Res proof** of a QBF, is there also a **short QRAT proof**?
 - **Short** = **polynomial** with respect to the size of the formula.
 - Our answer: **Yes!**

Simulating LQ-Res With QRAT

- How to show that there is a short QRAT proof for every short LQ-Res proof?

Simulating LQ-Res With QRAT

- How to show that there is a short QRAT proof for every short LQ-Res proof?
- ➔ Answer: With a simulation procedure.
 - Takes as input an LQ-Res proof and transforms it into a short QRAT proof.



Proving Unsatisfiability of QBFs: Long-Distance Resolution

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- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C} \text{ (\forall-red)} \qquad \frac{C \vee I \quad D \vee \bar{I}}{C \vee D} \text{ (LQ-Res)}$$

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Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Example proof with long-distance resolution:

$$\phi = \exists e_1 \forall u_1 \exists e_2 \exists e_3. (\bar{e}_1 \vee \bar{u}_1 \vee e_3) \wedge (\bar{u}_1 \vee e_2 \vee \bar{e}_3) \wedge (e_1 \vee u_1 \vee e_2) \wedge (\bar{e}_2)$$

$$\begin{array}{c}
 \frac{\frac{\frac{e_1 \vee u_1 \vee e_2}{u_1 \vee \bar{u}_1 \vee e_2} \quad \frac{\bar{u}_1 \vee e_2 \vee \bar{e}_3 \quad \bar{e}_1 \vee \bar{u}_1 \vee e_3}{\bar{e}_1 \vee \bar{u}_1 \vee e_2} \text{ (LQ-res)}}{u_1 \vee \bar{u}_1 \vee e_2} \quad \bar{e}_2 \text{ (LQ-res)}}{\frac{u_1 \vee \bar{u}_1}{u_1} \text{ (\forall-red)}} \text{ (LQ-res)} \\
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 \end{array}$$

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 - \forall -reduction of non-complementary literals
 - Blocked-literal elimination
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 - \forall -reduction of non-complementary literals
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Example: QRAT proof

1. $a_n \vee \bar{x}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (Q-res)
 2. $b_n \vee x_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (Q-res)
 3. $a_{n-1} \vee \bar{x}_{n-1} \vee \bar{b}_n \vee \bar{x}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (Q-res)
 4. $b_{n-1} \vee x_{n-1} \vee \bar{a}_n \vee x_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (Q-res)
 5. $a_{n-1} \vee \bar{x}_{n-1} \vee \bar{b}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (BLE of \bar{x}_n from 3)
 6. $b_{n-1} \vee x_{n-1} \vee \bar{a}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$ (BLE of x_n from 4)
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- ⋮

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on **outer resolvents**:
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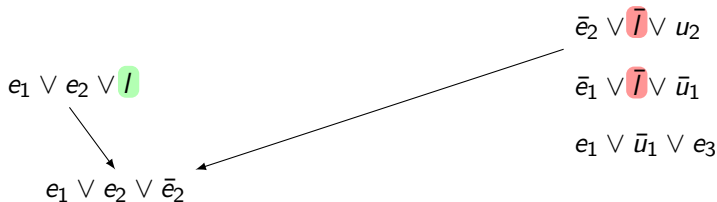
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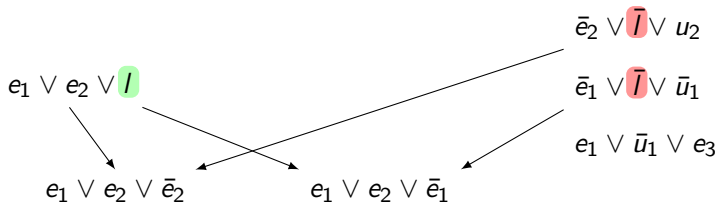
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- Problem: \forall -red of QRAT cannot remove **complementary literals**:

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 - But, using **blocked-literal addition**, we can always remove complementary literals.
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- By **successively removing complementary literals** from resolution steps, we obtain a valid QRAT proof.

Simulation Procedure: Results

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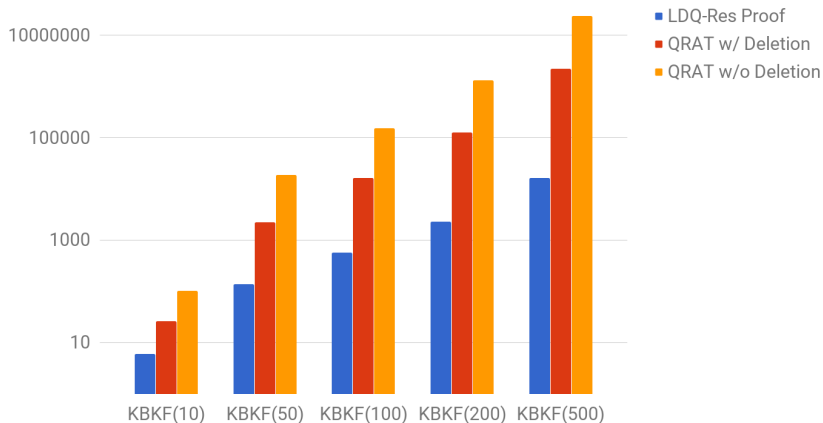
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- The tool allows to merge a QRAT proof of a preprocessor with a long-distance-resolution proof of a search-based solver.

Kleine Bünig Formulas (KBKF): LDQ-Res to QRAT

File size of generated proofs: LDQ-Res (Egly et al. 2013) to QRAT with and without deletion.



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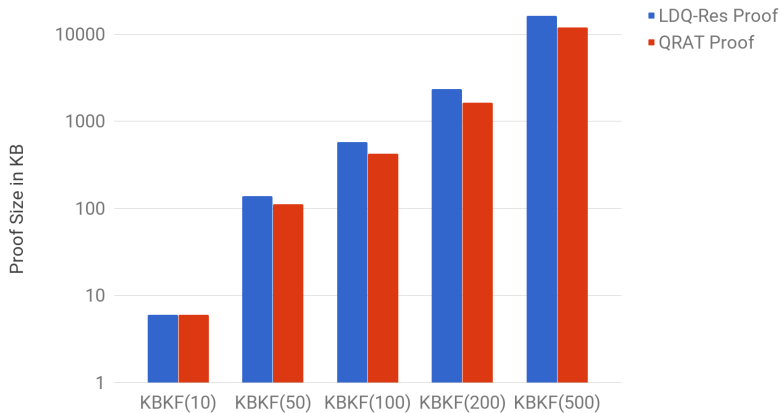
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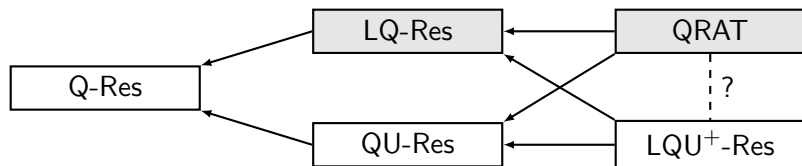
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Kleine Büning Formulas (KBKF): QRAT vs. LDQ-Res

File size of hand-crafted proofs: LDQ-Res (Egly et al. 2013) vs. QRAT.

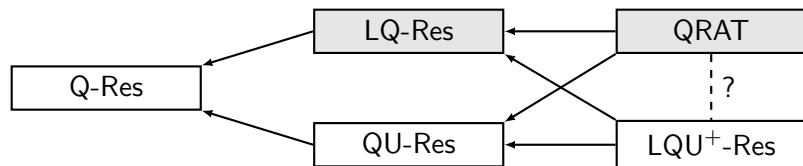


Complexity Landscape: QRAT and Resolution Systems



- **Open question:** Can QRAT simulate LQU⁺-Res?
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 - LQ-Res is a popular system for QBF **solving**.
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- Our **tool** allows to transform LQ-Res proofs into QRAT proofs.